Clifford algebras and Lie groups

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Contents

CHAPTER 1

Symmetric bilinear forms

In this section, we will describe the foundations of the theory of non-degenerate symmetric bilinear forms on finite-dimensional vector spaces, and their orthogonal groups. Among the highlights of this Chapter are the *Cartan-Dieudonné theorem*, which states that any orthogonal transformation is a finite product of reflections, and *Witt's theorem* giving a partial normal form for quadratic forms. The theory of *split* symmetric bilinear forms is found to have many parallels to the theory of symplectic forms, and we will give a discussion of the Lagrangian Grassmannian for this case. Throughout, $\mathbb{K}$ will denote a ground field of characteristic $\neq 2$. We are mainly interested in the cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and sometimes specialize to those two cases.

1. Quadratic vector spaces

Suppose V is a finite-dimensional vector space over $\mathbb{K}$. For any bilinear form $B\colon V \times V \to \mathbb{K}$, define a linear map

$$B^\flat\colon V \to V^*,\ v \mapsto B(v, \cdot).$$

The bilinear form B is called *symmetric* if it satisfies $B(v_1, v_2) = B(v_2, v_1)$ for all $v_1, v_2 \in V$. Since $\dim V < \infty$ this is equivalent to $(B^\flat)^* = B^\flat$. The symmetric bilinear form B is uniquely determined by the associated quadratic form, $Q_B(v) = B(v, v)$ by the *polarization identity*,

$$B(v, w) = \tfrac{1}{2}\big(Q_B(v + w) - Q_B(v) - Q_B(w)\big). \tag{1}$$

The *kernel* (also called *radical*) of B is the subspace

$$\ker(B) = \{v \in V \mid B(v, v_1) = 0 \text{ for all } v_1 \in V\},$$

i.e. the kernel of the linear map $B^\flat$. The bilinear form B is called *non-degenerate* if $\ker(B) = 0$, i.e. if and only if $B^\flat$ is an isomorphism. A vector space V together with a non-degenerate symmetric bilinear form B will be referred to as a *quadratic vector space*. Assume for the rest of this chapter that (V, B) is a quadratic vector space.

DEFINITION 1.1. A vector $v \in V$ is called *isotropic* if $B(v, v) = 0$, and *non-isotropic* if $B(v, v) \neq 0$.

For instance, if $V = \mathbb{C}^n$ over $\mathbb{K} = \mathbb{C}$, with the standard bilinear form $B(z, w) = \sum_{i=1}^n z_i w_i$, then $v = (1, i, 0, \ldots, 0)$ is an isotropic vector. If

$V = \mathbb{R}^2$ over $\mathbb{K} = \mathbb{R}$, with bilinear form $B(x, y) = x_1 y_1 - x_2 y_2$, then the set of isotropic vectors $x = (x_1, x_2)$ is given by the 'light cone' $x_1 = \pm x_2$.

The *orthogonal group* $\mathrm{O}(V)$ is the group

$$\text{(2)} \qquad \mathrm{O}(V) = \{A \in \mathrm{GL}(V) |\; B(Av, Aw) = B(v, w) \text{ for all } v, w \in V\}.$$

The subgroup of orthogonal transformations of determinant 1 is denoted $\mathrm{SO}(V)$, and is called the *special orthogonal group*.

For any subspace $F \subset V$, the *orthogonal* or *perpendicular* subspace is defined as

$$F^\perp = \{v \in V |\; B(v, v_1) = 0 \text{ for all } v_1 \in F\}.$$

The image of $B^\flat(F^\perp) \subset V^*$ is the annihilator of F. From this one deduces the dimension formula

$$\text{(3)} \qquad \dim F + \dim F^\perp = \dim V$$

and the identities

$$(F^\perp)^\perp = F, \quad (F_1 \cap F_2)^\perp = F_1^\perp + F_2^\perp, \quad (F_1 + F_2)^\perp = F_1^\perp \cap F_2^\perp$$

for all $F, F_1, F_2 \subset V$. For any subspace $F \subset V$ the restriction of B to F has kernel $\ker(B|_{F \times F}) = F \cap F^\perp$.

Definition 1.2. A subspace $F \subset V$ is called a *quadratic subspace* if the restriction of B to F is non-degenerate, that is $F \cap F^\perp = 0$.

Using $(F^\perp)^\perp = F$ we see that F is quadratic $\Leftrightarrow$ $F^\perp$ is quadratic $\Leftrightarrow$ $F \oplus F^\perp = V$.

As a simple application, one finds that any non-degenerate symmetric bilinear form B on V can be 'diagonalized'. Let us call a basis $E_1, \ldots, E_n$ of V an *orthogonal basis* if $B(E_i, E_j) = 0$ for all $i \neq j$.

Proposition 1.3. *Any quadratic vector space (V, B) admits an orthogonal basis $E_1, \ldots, E_n$. If $\mathbb{K} = \mathbb{C}$ one can arrange that $B(E_i, E_i) = 1$ for all i. If $\mathbb{K} = \mathbb{R}$ or $K = \mathbb{Q}$, one can arrange that $B(E_i, E_i) = \pm 1$ for all i.*

Proof. The proof is by induction on $n = \dim V$, the case $\dim V = 1$ being obvious. If $n > 1$ choose any non-isotropic vector $E_1 \in V$. The span of E_1 is a quadratic subspace, hence so is $\mathrm{span}(E_1)^\perp$. By induction, there is an orthogonal basis $E_2, \ldots, E_n$ of $\mathrm{span}(E_1)^\perp$. If $\mathbb{K} = \mathbb{C}$ (resp. $\mathbb{K} = \mathbb{R}, \mathbb{Q}$), one can rescale the E_i such that $B(E_i, E_i) = 1$ (resp. $B(E_i, E_i) = \pm 1$). $\square$

We will denote by $\mathbb{K}^{n,m}$ the vector space $\mathbb{K}^{n+m}$ with bilinear form given by $B(E_i, E_j) = \pm \delta_{ij}$, with a $+$ sign for $i = 1, \ldots, n$ and a $-$ sign for $i = n+1, \ldots, n+m$. If $m = 0$ we simple write $\mathbb{K}^n = \mathbb{K}^{n,0}$, and refer to the bilinear form as *standard*. The Proposition above shows that for $\mathbb{K} = \mathbb{C}$, and quadratic vector space (V, B) is isomorphic to $\mathbb{C}^n$ with the standard bilinear form, while for $\mathbb{K} = \mathbb{R}$ it is isomorphic to some $\mathbb{R}^{n,m}$. (Here n, m are uniquely determined, although it is not entirely obvious at this point.)

2. Isotropic subspaces

Let (V, B) be a quadratic vector space.

DEFINITION 2.1. A subspace $F \subset V$ is called *isotropic*[1] if $B|_{F \times F} = 0$, that is $F \subset F^\perp$.

The polarization identity (1) shows that a subspace $F \subset V$ is isotropic if and only if all of its vectors are isotropic. If $F \subset V$ is isotropic, then

$$\dim F \le \dim V/2 \tag{4}$$

since $\dim V = \dim F + \dim F^\perp \ge 2 \dim F$.

PROPOSITION 2.2. *For isotropic subspaces F, F' the following three conditions*

(a) *$F + F'$ is quadratic,*
(b) *$V = F \oplus (F')^\perp$,*
(c) *$V = F' \oplus F^\perp$*

are equivalent, and imply that $\dim F = \dim F'$. Given an isotropic subspace $F \subset V$ one can always find an isotropic subspace F' satisfying these conditions.

PROOF. We have

$$\begin{aligned}(F + F') \cap (F + F')^\perp &= (F + F') \cap F^\perp \cap (F')^\perp \\ &= (F + (F' \cap F^\perp)) \cap (F')^\perp \\ &= (F \cap (F')^\perp) + (F' \cap F^\perp).\end{aligned}$$

Thus

$$\begin{aligned}(F + F') \cap (F + F')^\perp = 0 &\Leftrightarrow F \cap (F')^\perp = 0 \text{ and } \ F' \cap F^\perp = 0 \\ &\Leftrightarrow F \cap (F')^\perp = 0, \text{ and } \ F + (F')^\perp = V.\end{aligned} \tag{5}$$

This shows (a)$\Leftrightarrow$(b), and similarly (a)$\Leftrightarrow$(c). Property (b) shows $\dim V = \dim F + (\dim F')^\perp = \dim F + \dim V - \dim F'$, hence $\dim F = \dim F'$. Given an isotropic subspace F, we find an isotropic subspace F' satisfying (c) as follows. Choose any complement W to $F^\perp$, so that

$$V = F^\perp \oplus W.$$

Thus $V = F^\perp + W$ and $0 = F^\perp \cap W$. Taking orthogonals, this is equivalent to $0 = F \cap W^\perp$ and $V = F + W^\perp$, that is

$$V = F \oplus W^\perp.$$

Let $S \colon W \to F \subset F^\perp$ be the projection along $W^\perp$. Then $w - S(w) \in W^\perp$ for all $w \in W$. The subspace

$$F' = \{w - \tfrac{1}{2} S(w) |\ w \in W\}.$$

[1] In some of the literature (e.g. C. Chevalley [**21**] or L. Grove [**32**]), a subspace is called isotropic if it contains at least one non-zero isotropic vector, and totally isotropic if all of its vectors are isotropic.

(being the graph of a map $W \to F^\perp$) is again a complement to $F^\perp$, and since for all $w \in$

$$B(w - \tfrac{1}{2}S(w), w - \tfrac{1}{2}S(w)) = B(w, w - S(w)) + \tfrac{1}{4}B(S(w), S(w)) = 0$$

(the first term vanishes since $w - S(w) \in W^\perp$, the second term vanishes since $S(w) \in F$ is isotropic) it follows that F' is isotropic. □

An isotropic subspace is called *maximal isotropic* if it is not properly contained in another isotropic subspace. Put differently, an isotropic subspace F is maximal isotropic if and only if it contains all $v \in F^\perp$ with $B(v, v) = 0$.

Proposition 2.3. *Suppose F, F' are maximal isotropic. Then*

(a) *the kernel of the restriction of B to $F + F'$ equals $F \cap F'$. (In particular, $F + F'$ is quadratic if and only if $F \cap F' = 0$.)*
(b) *The images of F, F' in the quadratic vector space $(F + F')/(F \cap F')$ are maximal isotropic.*
(c) $\dim F = \dim F'$.

Proof. Since F is maximal isotropic, it contains all isotropic vectors of $F^\perp$, and in particular it contains $F^\perp \cap F'$. Thus

$$F^\perp \cap F' = F \cap F'$$

Similarly $F \cap (F')^\perp = F \cap F'$ since F' is maximal isotropic. The calculation (5) hence shows

$$(F + F') \cap (F + F')^\perp = F \cap F',$$

proving (a). Let $W = (F + F')/(F \cap F')$ with the bilinear form B_W induced from B, and $\pi \colon F + F' \to W$ the quotient map. Clearly, B_W is non-degenerate, and $\pi(F), \pi(F')$ are isotropic. Hence the sum $W = \pi(F) + \pi(F')$ is a direct sum, and the two subspaces are maximal isotropic of dimension $\frac{1}{2}\dim W$. It follows that $\dim F = \dim \pi(F) + \dim(F \cap F') = \dim \pi(F') + \dim(F \cap F') = \dim F'$. □

Definition 2.4. The *Witt index* of a non-degenerate symmetric bilinear form B is the dimension of a maximal isotropic subspace.

By (4), the maximal Witt index is $\frac{1}{2}\dim V$ if $\dim V$ is even, and $\frac{1}{2}(\dim V - 1)$ if $\dim V$ is odd.

3. Split bilinear forms

Definition 3.1. The non-degenerate symmetric bilinear form B on an even-dimensional vector space V is called *split* if its Witt index is $\frac{1}{2}\dim V$. In this case, maximal isotropic subspaces are also called *Lagrangian subspaces.*

Equivalently, the Lagrangian subspaces are characterized by the property

$$F = F^\perp.$$

Let $\mathrm{Lag}(V)$ denote the *Lagrangian Grassmannian*, i.e. the set of Lagrangian subspaces of V.

PROPOSITION 3.2. *Suppose V is a vector space with split bilinear form, and F a Lagrangian subspace.*

(1) *The set of subspaces $R \subset V$ complementary to F carries a canonical affine structure, with $F \otimes F$ as its space of translations.*
(2) *The set of Lagrangian subspaces of V complementary to F carries a canonical affine structure, with $\wedge^2 F$ as its space of translations.*
(3) *If R is complementary to F, then so is $R^\perp$. The map $R \mapsto R^\perp$ an affine-linear involution on the set of complements, with fixed points the affine subspace of Lagrangian complements.*
(4) *For any complement R, the mid-point of the line segment between $R, R^\perp$ is a Lagrangian complement.*

PROOF. Let $\pi\colon V \to V/F$ be the quotient map. Define an injective group homomorphism

$$\mathrm{Hom}(V/F, F) \to \mathrm{GL}(V), \quad f \mapsto A_f$$

where $A_f(v) = v + f(\pi(v))$. It defines an action of the group $\mathrm{Hom}(V/F, F)$ on V, hence on the set of subspaces of V, preserving the set of subspaces complementary to F. The latter action is free and transitive. (Given two complements R, R', one can think of $R' \subset R \oplus F$ as the graph of a linear map $V/F \cong R \to F$.) This proves (1), since we may write

$$\mathrm{Hom}(V/F, F) = F \otimes (V/F)^* = F \otimes F$$

(using the bilinear form to identify $(V/F)^* = F$). If R is a complement to F, then $R^\perp$ is a complement to $F^\perp = F$. For (2), we similarly consider an inclusion

$$\wedge^2(F) \to \mathrm{O}(V), \ \phi \mapsto A_\phi$$

where $A_\phi(v) = v + \iota(\pi(v))\phi$ (identifying $V/F = F^*$). This subgroup of $\mathrm{O}(V)$ acts freely and transitively on the set of Lagrangian subspaces transverse to F: One checks that if F' is any Lagrangian complement to F, then all other Lagrangian complements are obtained as graphs of skew-symmetric linear maps $F^* = F' \to F$. This proves (2). For (3), observe first that if R is a complement to F, then $R^\perp$ is a complement to $F^\perp = F$. Obviously, the fixed point set of this involution are the Lagrangian complements. If $f \in \mathrm{Hom}(V/F, F)$, then $\langle f(\pi(v)), w\rangle = \langle v, f(\pi(w))\rangle$ from which one deduces that $A_f(R^\perp) = A_f(R)^\perp$. Hence the involution is affine-linear. For (4), we use that the involution preserves the line through $R, R^\perp$ and exchanges $R, R^\perp$. Hence it fixes the mid-point, which is therefore Lagrangian. □

REMARK 3.3. The construction in (4) can also be phrased more directly, as follows. Let R be a complement to F. Let $S\colon R \to F$ be the map defined by $\langle v, S(w)\rangle = \langle v, w\rangle$ for all $v, w \in R$. Then $R^\perp = \{v - S(v) |\ v \in R\}$, and the midpoint between $R, R^\perp$ is $\{v - \frac{1}{2}S(v) |\ v \in R\}$. Note that this is the construction used in the proof of Proposition 2.2.

PROPOSITION 3.4. *Let $F \subset V$ be a Lagrangian subspace. The group of orthogonal transformations of V fixing all points of F is the additive group $\wedge^2(F)$, embedded into $\mathrm{O}(V)$ by the map*

$$\phi \mapsto A_\phi, \quad A_\phi(v) = v + \iota(\pi(v))\phi.$$

(Here $\pi \colon V \to V/F \cong F^*$ is the projection.) The group $\mathrm{O}(V)_F$ of orthogonal transformations $A \in \mathrm{O}(V)$ taking F to itself is an extension

$$1 \to \wedge^2(F) \to \mathrm{O}(V)_F \to \mathrm{GL}(F) \to 1.$$

One has $\mathrm{O}(V)_F \subset \mathrm{SO}(V)$.

PROOF. It is clear that the subgroup $\wedge^2(F)$ fixes all points in F. Conversely, suppose A fixes all points of F. Fix a Lagrangian complement F' to F. Then $A(F')$ is again a complement to F, hence is related to F' by some $\phi \in \wedge^2(F)$. The transformation $\tilde{A} = A_\phi^{-1} \circ A$ preserves F'. For all $v \in F$ and $w \in F'$, we have $\langle v, \tilde{A}w\rangle = \langle \tilde{A}^{-1}v, w\rangle = \langle v, w\rangle$, hence $\tilde{A}w = w$. This shows that $\tilde{A}$ fixes all points of F, F', and is hence equal to the identity. That is, $A = A_\phi$. For any $A \in \mathrm{O}(V)_F$, the restriction to F defines an element of $\mathrm{GL}(F)$, which is trivial if and only if A fixes F pointwise, i.e. if lies in the subgroup $\wedge^2(F)$. The map to $\mathrm{GL}(F)$ is surjective: Given any $g \in \mathrm{GL}(F)$, let $F' \cong F^*$ carry the transformation $(g^{-1})^*$. The transformation $A = (g^{-1})^* \oplus g$ of $F^* \oplus F = F$ is orthogonal, and restricts to g on F. Note that this transformation has determinant 1, as do all transformations in $\wedge^2(F)$. $\square$

If F is a Lagrangian subspace, the choice of a Lagrangian complement $F' \cong F^*$ identifies V with $F^* \oplus F$, with the quadratic form given by the pairing:

$$B((\mu, v), (\mu, v)) = \langle \mu, v\rangle.$$

That is, $B((\mu_1, v_1), (\mu_2, v_2)) = \frac{1}{2}(\langle \mu_1, v_2\rangle + \langle \mu_2, v_1\rangle)$. Given such a Lagrangian splitting of V one can construct an adapted basis:

PROPOSITION 3.5. *Let (V, B) be a quadratic vector space with a split bilinear form. Then there exists a basis $e_1, \ldots, e_k, f_1, \ldots, f_k$ of V in which the bilinear form is given as follows:*

$$B(e_i, e_j) = 0, \; B(e_i, f_j) = \tfrac{1}{2}\delta_{ij}, \; B(f_i, f_j) = 0. \tag{6}$$

PROOF. Choose a pair of complementary Lagrangian subspaces, F, F'. Since B defines a non-degenerate pairing between F and F', it defines an isomorphism, $F' \cong F^*$. Choose a basis $e_1, \ldots, e_k$, and define $f_1, \ldots, f_k \in F'$ by $B(e_i, f_j) = \frac{1}{2}\delta_{ij}$. It is automatic that $B(e_i, e_j) = B(f_i, f_j) = 0$ since F, F' are Lagrangian. $\square$

Our basis $e_1, \ldots, e_k, f_1, \ldots, f_k$ for a quadratic vector space (V, B) with split bilinear form is not orthogonal. However, it may be replaced by an orthogonal basis

$$E_i = e_i + f_i, \quad \tilde{E}_i = e_i - f_i.$$

In the new basis, the bilinear form reads,

$$B(E_i, E_j) = \delta_{ij},\ B(E_i, \tilde{E}_j) = 0,\ B(\tilde{E}_i, \tilde{E}_j) = -\delta_{ij}. \tag{7}$$

The orthogonal group of $F^* \oplus F$ will be discussed in detail in §4, Section 2.2 below. At this point, let us rephrase Proposition 3.4 in terms of the splitting:

LEMMA 3.6. *The subgroup of orthogonal transformations fixing all points of $F \subset F^* \oplus F$ consists of all transformations of the form*

$$A_D \colon (\mu, v) \mapsto (\mu, v + D\mu)$$

where $D \colon F^* \to F$ is skew-adjoint: $D^* = -D$.

4. E.Cartan-Dieudonné's Theorem

Throughout this Section, we assume that (V, B) is a quadratic vector space. The following simple result will be frequently used.

LEMMA 4.1. *For any $A \in \mathrm{O}(V)$, the orthogonal of the space of A-fixed vectors equals the range of $A - I$:*

$$\mathrm{ran}(A - I) = \ker(A - I)^\perp.$$

PROOF. For any $L \in \mathrm{End}(V)$, the transpose $L^\top$ relative to B satisfies $\mathrm{ran}(L) = \ker(L^\top)^\perp$. We apply this to $L = A - I$, and observe that $\ker(A^\top - I) = \ker(A - I)$ since a vector is fixed under A if and only if it is fixed under $A^\top = A^{-1}$. $\square$

DEFINITION 4.2. An orthogonal transformation $R \in \mathrm{O}(V)$ is called a *reflection* if its fixed point set $\ker(R - I)$ has codimension 1.

Equivalently, $\mathrm{ran}(R - I) = \ker(R - I)^\perp$ is 1-dimensional. If $v \in V$ is a non-isotropic vector, then the formula

$$R_v(w) = w - 2\frac{B(v, w)}{B(v, v)}v,$$

defines a reflection, since $\mathrm{ran}(R_v - I) = \mathrm{span}(v)$ is 1-dimensional.

PROPOSITION 4.3. *Any reflection R is of the form R_v, where the non-isotropic vector v is unique up to a non-zero scalar.*

PROOF. Suppose R is a reflection, and consider the 1-dimensional subspace $F = \mathrm{ran}(R - I)$. We claim that F is a quadratic subspace of V. Once this is established, we obtain $R = R_v$ for any non-zero $v \in F$, since R_v then acts as -1 on F and as $+1$ on $F^\perp$. To prove the claim, suppose on the contrary that F is not quadratic. Since $\dim F = 1$ it is then isotropic. Let F' be an isotropic subspace such that $F + F'$ is quadratic. Since R fixes $(F + F')^\perp \subset F^\perp = \ker(R - I)$, it may be regarded as a reflection of $F + F'$. This reduces the problem to the case $\dim V = 2$, with $F \subset V$ maximal isotropic, and R fixes F pointwise. As we had seen, the group of such transformations are identified with the additive group of skew-symmetric maps

$F^* \to F$, but for $\dim F = 1$ this group is trivial. Hence R is the identity, contradicting $\dim \operatorname{ran}(R - I) = 1$. □

Some easy properties of reflections are,

(1) $\det(R) = -1$,
(2) $R^2 = I$,
(3) if v is non-isotropic, $AR_vA^{-1} = R_{Av}$ for all $A \in \mathrm{O}(V)$,
(4) distinct reflections $R_1 \neq R_2$ commute if and only if the lines $\operatorname{ran}(R_1 - I)$ and $\operatorname{ran}(R_2 - I)$ are orthogonal.

The last Property may be seen as follows: suppose $R_1R_2 = R_2R_1$ and apply to $v_1 \in \operatorname{ran}(R_1 - I)$. Then $R_1(R_2v_1) = -R_2v_1$, which implies that R_2v_1 is a multiple of v_1; in fact $R_2v_1 = \pm v_1$ since R_2 is orthogonal. Since $R_2v_1 = -v_1$ would imply that $R_1 = R_2$, we must have $R_2v_1 = v_1$, or $v_1 \in \ker(R_2 - I)$.

For any $A \in \mathrm{O}(V)$, let $l(A)$ denote the smallest number l such that $A = R_1 \cdots R_l$ where $R_i \in \mathrm{O}(V)$ are reflections. We put $l(I) = 0$, and for the time being we put $l(A) = \infty$ if A cannot be written as such a product. (The Cartan-Dieudonne theorem below states that $l(A) < \infty$ always.) The following properties are easily obtained from the definition, for all $A, g, A_1, A_2 \in \mathrm{O}(V)$,

$$\begin{aligned} l(A^{-1}) &= l(A), \\ l(gAg^{-1}) &= l(A), \\ |l(A_1) - l(A_2)| \leq l(A_1A_2) &\leq l(A_1) + l(A_2), \\ \det(A) &= (-1)^{l(A)} \end{aligned}$$

A little less obvious is the following estimate.

PROPOSITION 4.4. *For any $A \in \mathrm{O}(V)$, the number $l(A)$ is bounded below by the codimension of the fixed point set:*

$$\dim(\operatorname{ran}(A - I)) \leq l(A).$$

PROOF. Let $n(A) = \dim(\operatorname{ran}(A - I))$. If $A_1, A_2 \in \mathrm{O}(V)$, we have

$$\ker(A_1A_2 - I) \supseteq \ker(A_1A_2 - I) \cap \ker(A_1 - I) = \ker(A_2 - I) \cap \ker(A_1 - I)$$

Taking orthogonals,

$$\operatorname{ran}(A_1A_2 - I) \subseteq \operatorname{ran}(A_2 - I) + \operatorname{ran}(A_1 - I)$$

which shows

$$n(A_1A_2) \leq n(A_1) + n(A_2).$$

Thus, if $A = R_1 \cdots R_l$ is a product of $l = l(A)$ reflections, we have

$$n(A) \leq n(R_1) + \ldots + n(R_l) = l(A).$$ □

The following upper bound for $l(A)$ is much more tricky:

THEOREM 4.5 (E.Cartan-Dieudonné). *Any orthogonal transformation $A \in \mathrm{O}(V)$ can be written as a product of $l(A) \leq \dim V$ reflections.*

Proof. By induction, we may assume that the Theorem is true for quadratic vector spaces of dimension $\leq \dim V - 1$. We will consider three cases.

Case 1: $\ker(A - I)$ is non-isotropic. Choose any non-isotropic vector $v \in \ker(A - I)$. Then A fixes the span of v and restricts to an orthogonal transformation A_1 of $V_1 = \mathrm{span}(v)^\perp$. Using the induction hypothesis, we obtain

$$l(A) = l(A_1) \leq \dim V - 1. \tag{8}$$

Case 2: $\mathrm{ran}(A - I)$ is non-isotropic. We claim:

(C) There exists a non-isotropic element $w \in V$ such that $v = (A-I)w$ is non-isotropic.

Given v, w as in (C), we may argue as follows. Since $v = (A - I)w$, and hence $(A + I)w \in \mathrm{span}(v)^\perp$, we have

$$R_v(A - I)w = -(A - I)w, \quad R_v(A + I)w = (A + I)w.$$

Adding and dividing by 2 we find $R_v A w = w$. Since w is non-isotropic, this shows that the kernel of $R_v A - I$ is non-isotropic. Equation (8) applied to the orthogonal transformation $R_v A$ shows $l(R_v A) \leq \dim V - 1$. Hence $l(A) \leq \dim V$. It remains to prove the claim (C). Suppose it is false, so that we have:

$(\neg C)$ The transformation $A - I$ takes the set of non-isotropic elements into the set of isotropic elements.

Let $v = (A - I)w$ be a non-isotropic element in $\mathrm{ran}(A - I)$. By $(\neg C)$ the element w is isotropic. The orthogonal space $\mathrm{span}(w)^\perp$ is non-isotropic for dimensional reasons, hence there exists a non-isotropic element w_1 with $B(w, w_1) = 0$. Then $w_1,\ w + w_1,\ w - w_1$ are all non-isotropic, and by $(\neg C)$ their images

$$v_1 = (A - I)w_1,\ v + v_1 = (A - I)(w + w_1),\ v - v_1 = (A - I)(w - w_1)$$

are isotropic. But then the polarization identity

$$Q_B(v) = \tfrac{1}{2}(Q_B(v + v_1) + Q_B(v - v_1)) - Q_B(v_1) = 0$$

shows that v is isotropic, a contradiction. This proves (C).

Case 3: Both $\ker(A - I)$ and $\mathrm{ran}(A - I)$ are isotropic. Since these two subspaces are orthogonal, it follows that they are equal, and are both Lagrangian. This reduces the problem to the case $V = F^* \oplus F$, where $F = \ker(A - I)$, that is A fixes F pointwise. By Lemma 3.6 this implies $\det(A) = 1$. Let R_v be *any* reflection, then $A_1 = R_v A \in \mathrm{O}(V)$ has $\det(A_1) = -1$. Hence $\ker(A_1 - I)$ and $\mathrm{ran}(A_1 - I)$ cannot be both isotropic, and by the first two cases $l(A_1) \leq \dim V = 2\dim F$. But since $\det(A_1) = -1$, $l(A_1)$ must be *odd*, hence $l(A_1) < \dim V$ and therefore $l(A) \leq \dim V$. $\square$

REMARK 4.6. Our proof of Cartan-Dieudonne's theorem is a small modification of Artin's proof in [**7**]. If $\mathrm{char}(\mathbb{K}) = 2$, there exist counterexamples to the statement of the Theorem. See Chevalley [**22**, page 83].

5. Witt's Theorem

The following result is of fundamental importance in the theory of quadratic forms.

THEOREM 5.1 (Witt's Theorem). *Suppose $F, \tilde{F}$ are subspaces of a quadratic vector space (V, B), such that there exists an isometric isomorphism $\phi\colon F \to \tilde{F}$, i.e. $B(\phi(v), \phi(w)) = B(v, w)$ for all $v, w \in F$. Then ϕ extends to an orthogonal transformation $A \in \mathrm{O}(V)$.*

PROOF. By induction, we may assume that the Theorem is true for quadratic vector spaces of dimension $\leq \dim V - 1$. We will consider two cases.

Case 1: F is non-isotropic. Let $v \in F$ be a non-isotropic vector, and let $\tilde{v} = \phi(v)$. Then $Q_B(v) = Q_B(\tilde{v}) \neq 0$, and $v + \tilde{v}$ and $v - \tilde{v}$ are orthogonal. The polarization identity $Q_B(v) + Q_B(\tilde{V}) = \frac{1}{2}(Q_B(v + \tilde{v}) + Q_B(v - \tilde{v}))$ show that are not both isotropic; say $w = v + \tilde{v}$ is non-isotropic. The reflection R_w satisfies

$$R_w(v + \tilde{v}) = -(v + \tilde{v}), \quad R_w(v - \tilde{v}) = v - \tilde{v}.$$

Adding, and dividing by 2 we find that $R_w(v) = -\tilde{v}$. Let $Q = R_w R_v$. Then Q is an orthogonal transformation with $Q(v) = \tilde{v} = \phi(v)$.

Replacing F with $F' = Q(F)$, v with $v' = Q(v)$ and ϕ with $\phi' = \phi \circ Q^{-1}$, we may thus assume that $F \cap \tilde{F}$ contains a non-isotropic vector v such that $\phi(v) = v$. Let

$$V_1 = \mathrm{span}(v)^{\perp}, \quad F_1 = F \cap V_1, \quad \tilde{F}_1 = \tilde{F} \cap V_1$$

and $\phi_1\colon F_1 \to \tilde{F}_1$ the restriction of ϕ. By induction, there exists an orthogonal transformation $A_1 \in \mathrm{O}(V_1)$ extending ϕ_1. Let $A \in \mathrm{O}(V)$ with $A(v) = v$ and $A|_{V_1} = A_1$; then A extends ϕ.

Case 2: F is isotropic. Let F' be an isotropic complement to $F^{\perp}$, and let $\tilde{F}'$ be an isotropic complement to $\tilde{F}^{\perp}$. The pairing given by B identifies $F' \cong F^*$ and $\tilde{F}' \cong \tilde{F}^*$. The isomorphism $\phi\colon F \to \tilde{F}$ extends to an isometry $\psi\colon F \oplus F' \to \tilde{F} \oplus \tilde{F}'$, given by $(\phi^{-1})^*$ on $F' \cong F^*$. By Case 1 above, ψ extends further to an orthogonal transformation of V. $\square$

Some direct consequences are:

(1) $\mathrm{O}(V)$ acts transitively on the set of isotropic subspaces of any given dimension.

(2) If $F, \tilde{F}$ are isometric, then so are $F^\perp, \tilde{F}^\perp$. Indeed, any orthogonal extension of an isometry $\phi\colon F \to \tilde{F}$ restricts to an isometry of their orthogonals.

(3) Suppose $F \subset V$ is a subspace isometric to $\mathbb{K}^n$, with standard bilinear form $B(E_i, E_j) = \delta_{ij}$, and F is maximal relative to this property. If $F' \subset V$ is isometric to $\mathbb{K}^{n'}$, then there exists an orthogonal transformation $A \in \mathrm{O}(V)$ with $F' \subset A(F)$. In particular, the dimension of such a subspace F is an invariant of (V, B).

A subspace $W \subset V$ of a quadratic vector space is called *anisotropic* if it does not contain isotropic vectors other than 0. In particular, W is a quadratic subspace.

PROPOSITION 5.2 (Witt decomposition). *Any quadratic vector space (V, B) admits a decomposition $V = F \oplus F' \oplus W$ where F, F' are maximal isotropic, W is anisotropic, and $W^\perp = F \oplus F'$. If $V = F_1 \oplus F_1' \oplus W_1$ is another such decomposition, then there exists $A \in \mathrm{O}(V)$ with $A(F) = F_1,\ A(F') = F_1',\ A(W) = W_1$.*

PROOF. To construct such a decomposition, let F be a maximal isotropic subspace, and F' an isotropic complement to $F^\perp$. Then $F \oplus F'$ is quadratic, hence so is $W = (F \oplus F')^\perp$. Since F is maximal isotropic, the subspace W cannot contain isotropic vectors other than 0. Hence W is anisotropic. Given another such decomposition $V = F_1 \oplus F_1' \oplus W_1$, choose an isomorphism $F \cong F_1$. As we had seen (e.g. in the proof of Witt's Theorem), this extends canonically to an isometry $\phi\colon F \oplus F' \to F_1 \oplus F_1'$. Witt's Theorem gives an extension of ϕ to an orthogonal transformation $A \in \mathrm{O}(V)$. It is automatic that A takes $W = (F \oplus F')^\perp$ to $W = (F_1 \oplus F_1')^\perp$. □

EXAMPLE 5.3. If $\mathbb{K} = \mathbb{R}$, the bilinear form on the anisotropic part of the Witt decomposition is either positive definite (i.e. $Q_B(v) > 0$ for non-zero $v \in W$) or negative definite (i.e. $Q_B(v) < 0$ for non-zero $v \in W$). By Proposition 1.3, any quadratic vector space (V, B) over $\mathbb{R}$ is isometric to $\mathbb{R}^{n,m}$ for some n, m. The Witt decomposition shows that n, m are uniquely determined by B. Indeed $\min(n, m)$ is the Witt index of B, while the sign of $n - m$ is given by the sign of Q_B on the anisotropic part.

6. Orthogonal groups for $\mathbb{K} = \mathbb{R}, \mathbb{C}$

In this Section we discuss the structure of the orthogonal group $\mathrm{O}(V)$ for quadratic vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Being a closed subgroup of $\mathrm{GL}(V)$, the orthogonal group $\mathrm{O}(V)$ is a Lie group. Recall that for a Lie subgroup $G \subset \mathrm{GL}(V)$, the corresponding Lie algebra $\mathfrak{g}$ is the subspace of all $\xi \in \mathrm{End}(V)$ with the property $\exp(t\xi) \in G$ for all $t \in \mathbb{K}$ (using the exponential map of matrices). We have:

PROPOSITION 6.1. *The Lie algebra of $\mathrm{O}(V)$ is given by*

$$\mathfrak{o}(V) = \{A \in \mathrm{End}(V) |\ \ B(Av, w) + B(v, Aw) = 0 \textit{ for all } v, w \in V\},$$

with bracket given by commutator.

Proof. Suppose $A \in \mathfrak{o}(V)$, so that $\exp(tA) \in \mathrm{O}(V)$ for all t. Taking the t-derivative of $B(\exp(tA)v, \exp(tA)w) = B(v,w)$ we obtain $B(Av,w) + B(v,Aw) = 0$ for all $v, w \in V$. Conversely, given $A \in \mathfrak{gl}(V)$ with $B(Av,w) + B(v,Aw) = 0$ for all $v, w \in V$ we have

$$\begin{aligned} B(\exp(tA)v, \exp(tA)w) &= \sum_{k,l=0}^{\infty} \frac{t^{k+l}}{k!l!} B(A^k v, A^l w) \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{t^k}{i!(k-i)!} B(A^i v, A^{k-i} w) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i=0}^{k} \binom{k}{i} B(A^i v, A^{k-i} w) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} B(v, A^k w) \sum_{i=0}^{k} (-1)^i \binom{k}{i} \\ &= B(v,w) \end{aligned}$$

since $\sum_{i=0}^{k} (-1)^i \binom{k}{i} = \delta_{k,0}$. □

Thus $A \in \mathfrak{o}(V)$ if and only if $B^\flat \circ A \colon V \to V^*$ is a skew-adjoint map. In particular

$$\dim_{\mathbb{K}} \mathfrak{o}(V) = N(N-1)/2$$

where $N = \dim V$.

Let us now first discuss the case $\mathbb{K} = \mathbb{R}$. We have shown that any quadratic vector space (V, B) over $\mathbb{R}$ is isometric to $\mathbb{R}^{n,m}$, for unique n, m. The corresponding orthogonal group will be denoted $\mathrm{O}(n,m)$, the special orthogonal group $\mathrm{SO}(n,m)$, and its identity component $\mathrm{SO}_0(n,m)$. The dimension of $\mathrm{O}(n,m)$ coincides with the dimension of its Lie algebra $\mathfrak{o}(n,m)$, $N(N-1)/2$ where $N = n+m$. If $m = 0$ we will write $\mathrm{O}(n) = \mathrm{O}(n,0)$ and $\mathrm{SO}(n) = \mathrm{SO}(n,0)$. These groups are compact, since they are closed subsets of the unit ball in $\mathrm{Mat}(n, \mathbb{R})$.

Lemma 6.2. *The groups* $\mathrm{SO}(n)$ *are connected for all* $n \geq 1$*, and have fundamental group* $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ *for* $n \geq 3$.

Proof. The defining action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ restricts to a transitive action on the unit sphere S^{n-1}, with stabilizer at $(0, \ldots, 0, 1)$ equal to $\mathrm{SO}(n-1)$. Hence, for $n \geq 2$ the Lie group $\mathrm{SO}(n)$ is the total space of a principal fiber bundle over S^{n-1}, with fiber $\mathrm{SO}(n-1)$. This shows by induction that $\mathrm{SO}(n)$ is connected. The long exact sequence of homotopy groups

$$\cdots \to \pi_2(S^{n-1}) \to \pi_1(\mathrm{SO}(n-1)) \to \pi_1(\mathrm{SO}(n)) \to \pi_1(S^{n-1})$$

shows furthermore that the map $\pi_1(\mathrm{SO}(n-1)) \to \pi_1(\mathrm{SO}(n))$ is an isomorphism for $n > 3$ (since $\pi_2(S^{n-1}) = 0$ in that case). But $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$, since $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{R}P(3) = S^3/\mathbb{Z}_2$ (see below). $\square$

The groups $\mathrm{SO}(3)$ and $\mathrm{SO}(4)$ have a well-known relation with the group $\mathrm{SU}(2)$ of complex 2×2-matrices X satisfying $X^\dagger = X^{-1}$ and $\det X = 1$. Recall that the center of $\mathrm{SU}(2)$ is $\mathbb{Z}_2 = \{+I, -I\}$.

Proposition 6.3. *There are isomorphisms of Lie groups,*

$$\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2, \quad \mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$$

where in the second equality the quotient is by the diagonal subgroup $\mathbb{Z}_2 \subset \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Consider the algebra of quaternions $\mathbb{H} \cong \mathbb{C}^2 \cong \mathbb{R}^4$,

$$\mathbb{H} = \left\{ X = \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}, \ z, w \in \mathbb{C} \right\}.$$

For any $X \in \mathbb{H}$ let $||X|| = (|z|^2 + |w|^2)^{\frac{1}{2}}$. Note that $X^\dagger X = X X^\dagger = ||X||^2\, I$ for all $X \in \mathbb{H}$. Define a symmetric $\mathbb{R}$-bilinear form on $\mathbb{H}$ by

$$B(X_1, X_2) = \tfrac{1}{2} \operatorname{tr}(X_1^\dagger X_2).$$

The identification $\mathbb{H} \cong \mathbb{R}^4$ takes this to the standard bilinear form on $\mathbb{R}^4$ since $B(X, X) = \frac{1}{2} ||X||^2 \operatorname{tr}(I) = ||X||^2$. The unit sphere $S^3 \subset \mathbb{H}$, characterized by $||X||^2 = 1$ is the group $\mathrm{SU}(2) = \{X|\ X^\dagger = X^{-1},\ \det(X) = 1\}$. Define an action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{H}$ by

$$(X_1, X_2) \cdot X = X_1 X X_2^{-1}.$$

This action preserves the bilinear form on $\mathbb{H} \cong \mathbb{R}^4$, and hence defines a homomorphism $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$. The kernel of this homomorphism is the finite subgroup $\{\pm(I, I)\} \cong \mathbb{Z}_2$. (Indeed, $X_1 X X_2^{-1} = X$ for all X implies in particular $X_1 = X X_2 X^{-1}$ for all invertible X. But this is only possible if $X_1 = X_2 = \pm I$.) Since $\dim \mathrm{SO}(4) = 6 = 2 \dim \mathrm{SU}(2)$, and since $\mathrm{SO}(4)$ is connected, this homomorphism must be onto. Thus $\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm(I, I)\}$.

Similarly, identify $\mathbb{R}^3 \cong \{X \in \mathbb{H}|\ \operatorname{tr}(X) = 0\} = \operatorname{span}(I)^\perp$. The conjugation action of $\mathrm{SU}(2)$ on $\mathbb{H}$ preserves this subspace; hence we obtain a group homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. The kernel of this homomorphism is $\mathbb{Z}_2 \cong \{\pm I\} \subset \mathrm{SU}(2)$. Since $\mathrm{SO}(3)$ is connected and $\dim \mathrm{SO}(3) = 3 = \dim \mathrm{SU}(2)$, it follows that $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm I\}$. $\square$

To study the more general groups $\mathrm{SO}(n, m)$ and $\mathrm{O}(n, m)$, we recall the polar decomposition of matrices. Let

$$\mathrm{Sym}(k) = \{A|\ A^\top = A\} \subset \mathfrak{gl}(k, \mathbb{R})$$

be the space of real symmetric $k \times k$-matrices, and $\mathrm{Sym}^+(k)$ its subspace of positive definite matrices. As is well-known, the exponential map for matrices restricts to a diffeomorphism,

$$\exp\colon \mathrm{Sym}(k) \to \mathrm{Sym}^+(k),$$

with inverse $\log\colon \mathrm{Sym}^+(k) \to \mathrm{Sym}(k)$. Furthermore, the map

$$\mathrm{O}(k) \times \mathrm{Sym}(k) \to \mathrm{GL}(k, \mathbb{R}),\ (O, X) \mapsto Oe^X$$

is a diffeomorphism. The inverse map

$$\mathrm{GL}(k, \mathbb{R}) \to \mathrm{O}(k) \times \mathrm{Sym}(k),\ \mapsto (A|A|^{-1}, \log|A|),$$

where $|A| = (A^\top A)^{1/2}$, is called the *polar decomposition* for $\mathrm{GL}(k, \mathbb{R})$. We will need the following simple observation:

LEMMA 6.4. *Suppose $X \in \mathrm{Sym}(k)$ is non-zero. Then the closed subgroup of $\mathrm{GL}(k, \mathbb{R})$ generated by e^X is non-compact.*

PROOF. Replacing X with $-X$ if necessary, we may assume $||e^X|| > 1$. But then $||e^{nX}|| = ||e^X||^n$ goes to ∞ for $n \to \infty$. $\square$

This shows that $\mathrm{O}(k)$ is a maximal compact subgroup of $\mathrm{GL}(k, \mathbb{R})$. The polar decomposition for $\mathrm{GL}(k, \mathbb{R})$ restricts to a polar decomposition for any closed subgroup G that is invariant under the involution $A \mapsto A^\top$. Let

$$K = G \cap \mathrm{O}(k, \mathbb{R}),\ P = G \cap \mathrm{Sym}^+(k),\ \mathfrak{p} = \mathfrak{g} \cap \mathrm{Sym}(k).$$

The diffeomorphism $\exp\colon \mathrm{Sym}(k) \to \mathrm{Sym}^+(k)$ restricts to a diffeomorphism $\exp\colon \mathfrak{p} \to P$, with inverse the restriction of log. Hence the polar decomposition for $\mathrm{GL}(k, \mathbb{R})$ restricts to a diffeomorphism

$$K \times \mathfrak{p} \to G$$

whose inverse is called the polar decomposition of G. (It is a special case of a *Cartan decomposition.*) Using Lemma 6.4, we see that K is a maximal compact subgroup of G. Since $\mathfrak{p}$ is just a vector space, K is a deformation retract of G.

We will now apply these considerations to $G = \mathrm{O}(n, m)$. Let B_0 be the standard bilinear form on $\mathbb{R}^{n+m}$, and define the endomorphism J by

$$B(v, w) = B_0(Jv, w).$$

Thus J acts as the identity on $\mathbb{R}^n \oplus 0$ and as minus the identity $0 \oplus \mathbb{R}^m$, and an endomorphism of $\mathbb{R}^{n+m}$ commutes with J if and only if it preserves the direct sum decomposition $\mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$. A matrix $A \in \mathrm{Mat}(n+m, \mathbb{R})$ lies in $\mathrm{O}(n, m)$ if and only if $A^\top JA = J$, where $\top$ denotes as before the usual transpose of matrices, i.e. the transpose relative to B_0 (not relative to B). Similarly $X \in \mathfrak{o}(n, m)$ if and only if $X^\top J + JX = 0$.

REMARK 6.5. In block form we have

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$$

For $A \in \mathrm{Mat}(n+m,\mathbb{R})$ in block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{9}$$

we have $A \in \mathrm{O}(n,m)$ if and only if

$$a^\top a = I + c^\top c, \quad d^\top d = I + b^\top b, \quad a^\top b = c^\top d. \tag{10}$$

Similarly, for $X \in \mathrm{Mat}(n+m,\mathbb{R})$, written in block form

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{11}$$

we have $X \in \mathfrak{o}(n,m)$ if and only if

$$\alpha^\top = -\alpha,\ \beta^\top = \gamma,\ \delta^\top = -\delta. \tag{12}$$

Since $\mathrm{O}(n,m)$ is invariant under $A \mapsto A^\top$, (and likewise for the special orthogonal group and its identity component) the polar decomposition applies. We find:

PROPOSITION 6.6. *Relative to the polar decomposition of* $\mathrm{GL}(n+m,\mathbb{R})$*, the maximal subgroups of*

$$G = \mathrm{O}(n,m),\ \mathrm{SO}(n,m),\ \mathrm{SO}_0(n,m),$$

are, respectively,

$$K = \mathrm{O}(n) \times \mathrm{O}(m),\ \mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(m)),\ \mathrm{SO}(n) \times \mathrm{SO}(m).$$

(Here $\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(m))$ *are elements of* $(\mathrm{O}(n) \times \mathrm{O}(m))$ *of determinant* 1*.) In all of these cases, the space* $\mathfrak{p}$ *in the Cartan decomposition is given by matrices of the form*

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x \\ x^\top & 0 \end{pmatrix} \right\}$$

where x *is an arbitrary* $n \times m$*-matrix.*

PROOF. We start with $G = \mathrm{O}(n,m)$. Elements in $K = G \cap \mathrm{O}(n+m)$ are characterized by $A^\top J A = J$ and $A^\top A = I$. The two conditions give $AJ = JA$, so that A is a block diagonal element of $\mathrm{O}(n+m)$. Hence $A \in \mathrm{O}(n) \times \mathrm{O}(m) \subset \mathrm{O}(n,m)$. This shows $K = \mathrm{O}(n) \times \mathrm{O}(m)$. Elements $X \in \mathfrak{p} = \mathfrak{o}(n,m) \cap \mathrm{Sym}(n+m)$ satisfy $X^\top J + JX = 0$ and $X^\top = X$, hence they are symmetric block off-diagonal matrices. This proves our characterization of $\mathfrak{p}$, and proves the polar decomposition for $\mathrm{O}(n,m)$. The polar decompositions for $\mathrm{SO}(n,m)$ is an immediate consequence, and the polar decomposition for $\mathrm{SO}_0(n,m)$ follows since $\mathrm{SO}(n) \times \mathrm{SO}(m)$ is the identity component of $\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(m))$. □

COROLLARY 6.7. *Unless* $n = 0$ *or* $m = 0$ *the group* $\mathrm{O}(n,m)$ *has four connected components and* $\mathrm{SO}(n,m)$ *has two connected components.*

We next describe the space $P = \exp(\mathfrak{p})$.

PROPOSITION 6.8. *The space $P = \exp(\mathfrak{p}) \subset G$ consists of matrices*

$$P = \left\{ \begin{pmatrix} (I + bb^\top)^{1/2} & b \\ b^\top & (I + b^\top b)^{1/2} \end{pmatrix} \right\}$$

where b ranges over all $n \times m$-matrices. In fact,

$$\log \begin{pmatrix} (I + bb^\top)^{1/2} & b \\ b^\top & (I + b^\top b)^{1/2} \end{pmatrix} = \begin{pmatrix} 0 & x \\ x^\top & 0 \end{pmatrix}$$

where x and b are related as follows,

$$b = \frac{\sinh(xx^\top)}{xx^\top} x, \quad x = \frac{\operatorname{arsinh}((bb^\top)^{1/2})}{(bb^\top)^{1/2}} b. \tag{13}$$

Note that $xx^\top$ (resp. $bb^\top$) need not be invertible. The quotient $\frac{\sinh(xx^\top)}{xx^\top}$ is to be interpreted as $f(xx^\top)$ where $f(z)$ is the entire holomorphic function $\frac{\sinh z}{z}$, and $f(xx^\top)$ is given in terms of the spectral theorem or equivalently in terms of the power series expansion of f.

PROOF. Let $X = \begin{pmatrix} 0 & x \\ x^\top & 0 \end{pmatrix}$. By induction on k,

$$X^{2k} = \begin{pmatrix} (xx^\top)^k & 0 \\ 0 & (x^\top x)^k \end{pmatrix}, \ X^{2k+1} = \begin{pmatrix} 0 & (xx^\top)^k x \\ x(x^\top x)^k & 0 \end{pmatrix}.$$

This gives

$$\exp(X) = \begin{pmatrix} \cosh(xx^\top) & \frac{\sinh(xx^\top)}{xx^\top} x \\ x \frac{\sinh(x^\top x)}{x^\top x} & \cosh(x^\top x) \end{pmatrix},$$

which is exactly the form of elements in P with $b = \frac{\sinh(xx^\top)}{xx^\top} x$. The equation $\cosh(xx^\top) = (1 + bb^\top)^{1/2}$ gives $\sinh(xx^\top) = (bb^\top)^{1/2}$. Plugging this into the formula for b, we obtain the second equation in (13). $\square$

For later reference, we mention one more simple fact about the orthogonal and special orthogonal groups. Let $\mathbb{Z}_2$ be the center of $\mathrm{GL}(n + m, \mathbb{R})$ consisting of $\pm I$.

PROPOSITION 6.9. *For all n, m, the center of the group $\mathrm{O}(n, m)$ is $\mathbb{Z}_2$. Except in the cases $(n, m) = (0, 2), (2, 0)$, the center of $\mathrm{SO}(n, m)$ is $\mathbb{Z}_2$ if $-I$ lies in $\mathrm{SO}(n, m)$, and is trivial otherwise. The statement for the identity component is similar.*

The proof is left as an exercise. (Note that the elements of the center of G commute in particular with the diagonal elements of G. In the case of hand, one uses this fact to argue that the central elements are themselves diagonal, and finally that they are multiples of the identity.)

The discussion above carries over to $\mathbb{K} = \mathbb{C}$, with only minor modifications. It is enough to consider the case $V = \mathbb{C}^n$, with the standard symmetric bilinear form. Again, our starting point is the polar decomposition, but now for complex matrices. Let $\mathrm{Herm}(n) = \{A | \ A^\dagger = A\}$ be the space of Hermitian $n \times n$ matrices, and $\mathrm{Herm}^+(n)$ the subset of positive definite matrices. The exponential map gives a diffeomorphism

$$\mathrm{Herm}(n) \to \mathrm{Herm}^+(n),\ X \mapsto e^X.$$

This is used to show that the map

$$\mathrm{U}(n) \times \mathrm{Herm}(n) \to \mathrm{GL}(n, \mathbb{C}), \quad (U, X) \mapsto Ue^X$$

is a diffeomorphism; the inverse map takes A to (Ae^{-X}, X) with $X = \frac{1}{2}\log(A^\dagger A)$. The polar decomposition of $\mathrm{GL}(n, \mathbb{C})$ gives rise to polar decompositions of any closed subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ that is invariant under the involution $\dagger$. In particular, this applies to $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$. Indeed, if $A \in \mathrm{O}(n, \mathbb{C})$, the matrix $A^\dagger A$ lies in $\mathrm{O}(n, \mathbb{C}) \cap \mathrm{Herm}(n)$, and hence its logarithm $X = \frac{1}{2}\log(A^\dagger A)$ lies in $\mathfrak{o}(n, \mathbb{C}) \cap \mathrm{Herm}(n)$. But clearly,

$$\mathrm{O}(n, \mathbb{C}) \cap \mathrm{U}(n) = \mathrm{O}(n, \mathbb{R}),$$
$$\mathrm{SO}(n, \mathbb{C}) \cap \mathrm{U}(n) = \mathrm{SO}(n, \mathbb{R})$$

while

$$\mathfrak{o}(n, \mathbb{C}) \cap \mathrm{Herm}(n) = \sqrt{-1}\mathfrak{o}(n, \mathbb{R}).$$

Hence, the maps $(U, X) \mapsto Ue^X$ restrict to polar decompositions

$$\mathrm{O}(n, \mathbb{R}) \times \sqrt{-1}\mathfrak{o}(n, \mathbb{R}) \to \mathrm{O}(n, \mathbb{C}),$$
$$\mathrm{SO}(n, \mathbb{R}) \times \sqrt{-1}\mathfrak{o}(n, \mathbb{R}) \to \mathrm{SO}(n, \mathbb{C}),$$

which shows that the algebraic topology of the complex orthogonal and special orthogonal group coincides with that of its real counterparts. Arguing as in the real case, the center of $\mathrm{O}(n, \mathbb{C})$ is given by $\{+I, -I\}$ while the center of $\mathrm{SO}(n, \mathbb{C})$ is trivial for n odd and $\{+I, -I\}$ for n even, provided $n \geq 3$.

7. Lagrangian Grassmannians

If (V, B) is a quadratic vector space with split bilinear form, denote by $\mathrm{Lag}(V)$ the set of Lagrangian subspaces. Recall that any such V is isomorphic to $\mathbb{K}^{n,n}$ where $\dim V = 2n$. For $\mathbb{K} = \mathbb{R}$ we have the following result.

THEOREM 7.1. *Let $V = \mathbb{R}^{n,n}$ with the standard basis satisfying* (7). *Then the maximal compact subgroup $\mathrm{O}(n) \times \mathrm{O}(n)$ of $\mathrm{O}(n, n)$ acts transitively on the space $\mathrm{Lag}(\mathbb{R}^{n,n})$ of Lagrangian subspaces, with stabilizer at*

$$L_0 = \mathrm{span}\{E_1 + \tilde{E}_1, \ldots, E_n + \tilde{E}_n\} \tag{14}$$

the diagonal subgroup $\mathrm{O}(n)_\Delta$. Thus

$$\mathrm{Lag}(\mathbb{R}^{n,n}) \cong \mathrm{O}(n) \times \mathrm{O}(n)/\,\mathrm{O}(n)_\Delta \cong \mathrm{O}(n).$$

In particular, it is a compact space with two connected components.

PROOF. Let B_0 be the standard positive definite bilinear form on the vector space $\mathbb{R}^{n,n} = \mathbb{R}^{2n}$, with corresponding orthogonal group $\mathrm{O}(2n)$. Introduce an involution $J \in \mathrm{O}(2n)$, by

$$B(v,w) = B_0(Jv,w).$$

That is $JE_i = E_i$, $J\tilde{E}_i = -\tilde{E}_i$. Then the maximal compact subgroup $\mathrm{O}(n) \times \mathrm{O}(n)$ consists of all those transformations $A \in \mathrm{O}(n,n)$ which commute with J. At the same time, $\mathrm{O}(n) \times \mathrm{O}(n)$ is characterized as the orthogonal transformations in $\mathrm{O}(2n)$ commuting with J.

The ± 1 eigenspaces $V_\pm$ of J are both anisotropic, i.e. they do not contain any isotropic vectors. Hence, if $L \subset \mathbb{R}^{n,n}$ is Lagrangian, then $J(L)$ is transverse to L:

$$L \cap J(L) = (L \cap V_+) \oplus (L \cap V_-) = 0.$$

For any L, we may choose a basis $v_1, \ldots, v_n$ that is orthonormal relative to B_0. Then $v_1, \ldots, v_n, J(v_1), \ldots, J(v_n)$ is a B_0-orthonormal basis of $\mathbb{R}^{n,n}$. If L' is another Lagrangian subspace, with B_0-orthonormal basis $v_1', \ldots, v_n'$, then the orthogonal transformation $A \in \mathrm{O}(2n)$ given by

$$Av_i = v_i', \quad AJ(v_i) = J(v_i'), \; i = 1, \ldots, n$$

commutes with J, hence $A \in \mathrm{O}(n) \times \mathrm{O}(n)$. This shows that $\mathrm{O}(n) \times \mathrm{O}(n)$ acts transitively on $\mathrm{Lag}(\mathbb{R}^{n,n})$. For the Lagrangian subspace (14), with $v_i = \frac{1}{\sqrt{2}}(E_i + \tilde{E}_i)$, the stabilizer of L_0 under the action of $\mathrm{O}(n) \times \mathrm{O}(n)$ consists of those transformations $A \in \mathrm{O}(n) \times \mathrm{O}(n)$ for which $v_1', \ldots, v_n'$ is again a B_0-orthonormal basis of L_0. But this is just the diagonal subgroup $\mathrm{O}(n)_\Delta \subset \mathrm{O}(n) \times \mathrm{O}(n)$. Finally, since the multiplication map

$$(\mathrm{O}(n) \times \{1\}) \times \mathrm{O}(n)_\Delta \to \mathrm{O}(n) \times \mathrm{O}(n)$$

is a bijection, the quotient is just $\mathrm{O}(n)$. $\square$

Theorem 7.1 does not, as it stands, hold for other fields $\mathbb{K}$. Indeed, the group $\mathrm{O}(n,\mathbb{K}) \times \mathrm{O}(n,\mathbb{K})$ takes L_0 to a Lagrangian subspace transverse to V_+, V_-. However, there may be other Lagrangian subspaces: E.g. if $\mathbb{K} = \mathbb{C}$ and $n = 2$, the span of $E_1 + \sqrt{-1}E_2$ and $\tilde{E}_1 + \sqrt{-1}\tilde{E}_2$ is a Lagrangian subspace not transverse to $V_\pm$. Nonetheless, there is a good description of the space Lag in the complex case $\mathbb{K} = \mathbb{C}$.

THEOREM 7.2. *The space of Lagrangian subspaces of $V = \mathbb{C}^{2m}$ is a homogeneous space*

$$\mathrm{Lag}(\mathbb{C}^{2m}) \cong \mathrm{O}(2m)/\,\mathrm{U}(m).$$

In particular, it is a compact space with two connected components.

PROOF. Let $v \mapsto \overline{v}$ be complex conjugation in $\mathbb{C}^{2m}$. Then $\langle v, w \rangle = B(\overline{v}, w)$ is the standard Hermitian inner product on $\mathbb{C}^{2m}$. Let $L_0 \subset \mathbb{C}^{2m}$ be the Lagrangian subspace spanned by

$$v_1 = \frac{1}{\sqrt{2}}(E_1 - \sqrt{-1}E_{m+1}), \ldots, v_m = \frac{1}{\sqrt{2}}(E_m - \sqrt{-1}E_{2m}),$$

where $E_1, \ldots, E_{2m}$ is the standard basis of $\mathbb{C}^{2m}$. Note that $v_1, \ldots, v_m$ is orthonormal for the Hermitian inner product, and $v_1, \ldots, v_m, \overline{v}_1, \ldots, \overline{v}_m$ is an orthonormal basis of $\mathbb{C}^{2m}$. If L' is another Lagrangian subspace, with orthonormal basis $v_1', \ldots, v_m' \in L'$, then the unitary transformation $A \in \mathrm{U}(2m)$ taking $v_i, \overline{v}_i$ to $v_i', \overline{v}_i'$ commutes with complex conjugation, hence it actually lies in $\mathrm{O}(2m)$. This shows that $\mathrm{O}(2m)$ acts transitively on $\mathrm{Lag}(\mathbb{C}^{2m})$. The transformations $A \in \mathrm{O}(2m) \subset \mathrm{U}(2m)$ preserving L_0 are those for which $v_i' = A(v_i)$ is again an orthonormal basis of L. Hence, the stabilizer of L is $\mathrm{U}(m) \subset \mathrm{O}(2m)$. $\square$

Remark 7.3. The orbit of L_0 under $\mathrm{O}(m, \mathbb{C}) \times \mathrm{O}(m, \mathbb{C})$ is open and dense in $\mathrm{Lag}(\mathbb{C}^{2m})$, and as in the real case is identified with $\mathrm{O}(m, \mathbb{C})$. Thus, $\mathrm{Lag}(\mathbb{C}^{2m})$ is a smooth compactification of the complex Lie group $\mathrm{O}(m, \mathbb{C})$.

Theorem 7.2 has a well-known geometric interpretation. View $\mathbb{C}^{2m}$ as the complexification of $\mathbb{R}^{2m}$. Recall that an *orthogonal complex structure* on $\mathbb{R}^{2m}$ is an automorphism $J \in \mathrm{O}(2m)$ with $J^2 = -I$. We denote by J_0 the standard complex structure.

Let $\mathcal{J}(2m)$ denote the space of all orthogonal complex structures. It carries a transitive action of $\mathrm{O}(2m)$, with stabilizer at J_0 equal to $\mathrm{U}(m)$. Hence the space of orthogonal complex structures is identified with the complex Lagrangian Grassmannian:

$$\mathcal{J}(2m) = \mathrm{O}(2m)/\,\mathrm{U}(m) = \mathrm{Lag}(\mathbb{C}^{2m}).$$

Explicitly, this correspondence takes $J \in \mathcal{J}(2m)$ to its $+\sqrt{-1}$ eigenspace

$$L = \ker(J - \sqrt{-1}I).$$

This has complex dimension m since $\mathbb{C}^{2m} = L \oplus \overline{L}$, and it is isotropic since $v \in L$ implies

$$B(v, v) = B(Jv, Jv) = B(\sqrt{-1}v, \sqrt{-1}v) = -B(v, v).$$

Any Lagrangian subspace L determines J, as follows: Given $w \in \mathbb{R}^{2n}$, we may uniquely write $w = v + \overline{v}$ where $v \in L$. Define a linear map J by $Jw := -2\,\mathrm{Im}(v)$. Then $v = w - \sqrt{-1}Jw$. Since L is Lagrangian, we have

$$\begin{aligned} 0 = B(v, v) &= B(w - \sqrt{-1}Jw, w - \sqrt{-1}Jw) \\ &= B(w, w) - B(Jw, Jw) - 2\sqrt{-1}B(w, Jw), \end{aligned}$$

which shows that $J \in \mathrm{O}(2m)$ and that $B(w, Jw) = 0$ for all w. Multiplying the definition of J by $\sqrt{-1}$, we get

$$\sqrt{-1}v = \sqrt{-1}w + Jw$$

which shows that $J(Jw) = -w$. Hence J is an orthogonal complex structure.

Remark 7.4. Let $\mathrm{Iso}_{\max}(\mathbb{R}^{n,m})$ denote the set of maximal isotropic subspaces of $\mathbb{R}^{n,m}$. Then $\mathrm{Iso}_{\max}(\mathbb{R}^{n,m}) \cong \mathrm{Lag}(\mathbb{R}^{m,n})$. For $n \leq m$ the space $\mathrm{Iso}_{\max}(\mathbb{R}^{n,m})$ is isomorphic to $\mathrm{O}(m)/\,\mathrm{O}(m-n)$.

REMARK 7.5. There are parallel results in symplectic geometry, for vector spaces V with a non-degenerate *skew*-symmetric linear form ω. If $\mathbb{K} = \mathbb{R}$, any such V is identified with $\mathbb{R}^{2n} = \mathbb{C}^n$ with the standard symplectic form, $L_0 = \mathbb{R}^n \subset \mathbb{C}^n$ is a Lagrangian subspace, and the action of $\mathrm{U}(n) \subset \mathrm{Sp}(V, \omega)$ on L_0 identifies

$$\mathrm{Lag}_\omega(\mathbb{R}^{2n}) \cong \mathrm{U}(n)/\,\mathrm{O}(n)$$

For the space $\mathrm{Lag}(V)$ of complex Lagrangian Grassmannian subspaces of the complex symplectic vector space $\mathbb{C}^{2n} \cong \mathbf{H}^n$ one has

$$\mathrm{Lag}_\omega(\mathbb{C}^{2n}) \cong \mathrm{Sp}(n)/U(n)$$

where $\mathrm{Sp}(n)$ is the *compact symplectic group* (i.e. the quaternionic unitary group). See e.g. [**28**, p.67].

CHAPTER 2

Clifford algebras

Associated to any vector space V with bilinear form B is a Clifford algebra $\mathrm{Cl}(V;B)$. In the special case $B=0$, the Clifford algebra is just the exterior algebra $\wedge(V)$, and in the general case the Cllifford algebra can be regarded as a deformation of the exterior algebra. In this Chapter, after constructing the Clifford algebra and describing its basic properties, we study in some detail the quantization map $q\colon \wedge(V) \to \mathrm{Cl}(V;B)$ and justify the term 'quantization'. Throughout, we assume that V is a finite-dimensional vector space over a field $\mathbb{K}$ of characteristic 0.

1. Exterior algebras

1.1. Definition. For any vector space V over a field $\mathbb{K}$, let $T(V) = \bigoplus_{k\in\mathbb{Z}} T^k(V)$ be the tensor algebra, with $T^k(V) = V\otimes\cdots\otimes V$ the k-fold tensor product. The quotient of $T(V)$ by the two-sided ideal $\mathcal{I}(V)$ generated by all $v\otimes w + w\otimes v$ is the exterior algebra, denoted $\wedge(V)$. The product in $\wedge(V)$ is usually denoted $\alpha_1\wedge\alpha_2$, although we will frequently omit the wedge symbol and just write $\alpha_1\alpha_2$. Since $\mathcal{I}(V)$ is a *graded* ideal, the exterior algebra inherits a grading

$$\wedge(V) = \bigoplus_{k\in\mathbb{Z}} \wedge^k(V)$$

where $\wedge^k(V)$ is the image of $T^k(V)$ under the quotient map. We will write $|\phi| = k$ if $\phi\in\wedge^k(V)$. Clearly, $\wedge^0(V) = \mathbb{K}$ and $\wedge^1(V) = V$ so that we can think of V as a subspace of $\wedge(V)$. We may thus think of $\wedge(V)$ as the associative algebra linearly generated by V, subject to the relations $v\wedge w + w\wedge v = 0$.

Throughout, we will regard $\wedge(V)$ as a *graded super algebra*, where the $\mathbb{Z}_2$-grading is the mod 2 reduction of the $\mathbb{Z}$-grading.[1] Thus tensor products, derivations, and other constructions with $\wedge(V)$ are all understood in the super sense, often without further specification. Since

$$[\phi_1,\phi_2] \equiv \phi_1\wedge\phi_2 + (-1)^{k_1k_2}\phi_2\wedge\phi_1 = 0$$

for $\phi_1\in\wedge^{k_1}(V)$ and $\phi_2\in\wedge^{k_2}(V)$, we see that $\wedge(V)$ is *commutative* (as a super algebra).

[1] We refer to Appendix A for terminology and background regarding super vector spaces.

If V has dimension n, with basis $e_1, \ldots, e_n$, the space $\wedge^k(V)$ has basis

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$$

for all ordered subsets $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$. (If $k = 0$, we put $e_\emptyset = 1$.) In particular, we see that $\dim \wedge^k(V) = \binom{n}{k}$, and

$$\dim \wedge(V) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Letting $e^i \in V^*$ denote the dual basis to the basis e_i considered above, we define a dual basis to e_I to be the basis $e^I = e^{i_1} \wedge \cdots \wedge e^{i_k} \in \wedge(V^*)$.

1.2. Universal property, functoriality. The exterior algebra is characterized among graded super algebras by its *universal property*: If $\mathcal{A}$ is a graded super algebra, and $f\colon V \to \mathcal{A}^1$ a linear map with $f(v)f(w) + f(w)f(v) = 0$ for all $v, w \in V$, then f extends uniquely to a morphism of graded super algebras $f_\wedge\colon \wedge(V) \to \mathcal{A}$. Similar universal properties characterize $\wedge(V)$ in the categories of algebras, super algebras, graded algebras, filtered algebras, or filtered super algebras.

Any linear map $L\colon V \to W$ extends uniquely (by the universal property, applied to L viewed as a map into $V \to \wedge(W)$) to a morphism of graded super algebras $\wedge(L)\colon \wedge(V) \to \wedge(W)$. One has

$$\wedge(L_1 \circ L_2) = \wedge(L_1) \circ \wedge(L_2), \quad \wedge(\mathrm{id}_V) = \mathrm{id}_{\wedge(V)}.$$

As a special case, taking L to be the zero map $0\colon V \to V$ the resulting algebra homomorphism $\wedge(0)$ is the *augmentation map* (taking $\phi \in \wedge(V)$ to its component in $\wedge^0(V) \cong \mathbb{K}$). Taking L to be the map $v \mapsto -v$, the map $\wedge(L)$ is the *parity homomorphism* $\Pi \in \mathrm{Aut}(\wedge(V))$, equal to $(-1)^k$ on $\wedge^k(V)$.

The functoriality gives an algebra homomorphism

$$\mathrm{End}(V) \to \mathrm{End}_{\mathrm{alg}}(\wedge(V)), \quad A \mapsto \wedge(A)$$

[2] and, by restriction to invertible elements, a group homomorphism

$$\mathrm{GL}(V) \to \mathrm{Aut}_{\mathrm{alg}}(\wedge(V)), \; g \mapsto \wedge(g)$$

into the group of degree preserving algebra automorphisms of $\wedge(V)$. We will often write g in place of $\wedge(g)$, but reserve this notation for invertible transformations since e.g. $\wedge(0) \neq 0$.

Suppose V_1, V_2 are two vector spaces. Then $\wedge(V_1) \otimes \wedge(V_2)$ (tensor product of graded super algebras) with the natural inclusion of $V_1 \oplus V_2$ satisfies the universal property of the exterior algebra over $V_1 \oplus V_2$. Hence the morphism of graded super algebras

$$\wedge(V_1 \oplus V_2) \to \wedge(V_1) \otimes \wedge(V_2)$$

[2] If $\mathcal{A}$ is any algebra, we denote by $\mathrm{End}(\mathcal{A})$ (resp. $\mathrm{Aut}(\mathcal{A})$) the vector space homomorphisms (res. automorphisms) $\mathcal{A} \to \mathcal{A}$, while $\mathrm{End}_{\mathrm{alg}}(\mathcal{A})$ (resp. $\mathrm{Aut}_{\mathrm{alg}}(V)$) denotes the set of algebra homomorphisms (resp. group of algebra automorphisms).

intertwining the two inclusions is an isomorphism. As a special case, $\wedge(\mathbb{K}^n) = \wedge(\mathbb{K})\otimes\cdots\otimes\wedge(\mathbb{K})$.

The space $\mathrm{Der}(\wedge V)$ of derivations of the graded super algebra $\wedge V$ is a left module over $\wedge(V)$, since $\wedge(V)$ is commutative. Any such derivation is uniquely determined by its restriction to the space $V\subset\wedge(V)$ of generators, and conversely any linear map $V\to\wedge(V)$ extends to a derivation. Thus

$$\mathrm{Der}(\wedge V)\cong\mathrm{Hom}(V,\wedge V)$$

as graded super vector spaces, where the grading on the right hand side is $\mathrm{Hom}^k(V,\wedge V)=\mathrm{Hom}(V,\wedge^{k+1}(V))$.

In particular, $\mathrm{Der}^k(\wedge V)$ vanishes if $k<-1$. Elements of the space $\mathrm{Der}^{-1}(\wedge V)=\mathrm{Hom}(V,\mathbb{K})=V^*$ are called *contractions*. Explicitly, the derivation $\iota(\alpha)$ corresponding to $\alpha\in V^*$ is given by $\iota(\alpha)1=0$ and

$$\iota(\alpha)(v_1\wedge\cdots v_k)=\sum_{i=1}^k(-1)^{i-1}\langle\alpha,v_i\rangle\; v_1\wedge\cdots\widehat{v_i}\cdots\wedge v_k \tag{15}$$

for $v_1,\ldots,v_k\in V$. The contraction operators satisfy $\iota(\alpha)\iota(\beta)+\iota(\beta)\iota(\alpha)=0$ for $\alpha,\beta\in V^*$. Hence the map $\iota\colon V^*\to\mathrm{End}(\wedge(V))$ extends, by the universal property, to a morphism of super algebras

$$\iota\colon\wedge(V^*)\to\mathrm{End}(\wedge V).$$

This map takes $\wedge^k(V^*)$ to $\mathrm{End}^{-k}(\wedge V)$, hence it becomes a morphism of graded super algebras if we use the opposite $\mathbb{Z}$-grading on $\wedge V^*$.

On the other hand, left multiplication defines a morphism of graded super algebras

$$\epsilon\colon\wedge(V)\to\mathrm{End}(\wedge(V)),$$

called exterior multiplication. The operators $\epsilon(v)$ for $v\in V$ and $\iota(\alpha)$ for $\alpha\in V^*$ satisfy commutator relations,

$$\begin{aligned}
{[\epsilon(v),\epsilon(w)]}&\equiv\epsilon(v)\epsilon(w)+\epsilon(w)\epsilon(v)=0,\\
{[\iota(\alpha),\iota(\beta)]}&\equiv\iota(\alpha)\iota(\beta)+\iota(\beta)\iota(\alpha)=0,\\
{[\iota(\alpha),\epsilon(v)]}&\equiv\iota(\alpha)\epsilon(v)+\epsilon(v)\iota(\alpha)=\langle\alpha,v\rangle.
\end{aligned}\tag{16}$$

For later reference, observe that

$$\ker(\iota(\alpha))=\mathrm{ran}(\iota(\alpha))=\wedge(\ker(\alpha))\subset\wedge(V)$$

for all $\alpha\in V^*$. (To see this, decompose V into $\ker(\alpha)$ an a complement V_1, and use that $\wedge(V)=\wedge(\ker(\alpha))\otimes\wedge(V_1)$.) Similarly, $\ker(\epsilon(v))=\mathrm{ran}(\epsilon(v))$ is the ideal generated by $\mathrm{span}(v)$.

2. Clifford algebras

Clifford algebras are a generalization of exterior algebras, defined in the presence of a symmetric bilinear form.

2.1. Definition and first properties. Let V be a vector space over $\mathbb{K}$, with a symmetric bilinear form $B\colon V \times V \to \mathbb{K}$ (possibly degenerate).

DEFINITION 2.1. The *Clifford algebra* $\mathrm{Cl}(V;B)$ is the quotient

$$\mathrm{Cl}(V;B) = T(V)/\mathcal{I}(V;B)$$

where $\mathcal{I}(V;B) \subset T(V)$ is the two-sided ideal generated by all elements of the form

$$v \otimes w + w \otimes v - 2B(v,w)1, \quad v,w \in V.$$

Clearly, $\mathrm{Cl}(V;0) = \wedge(V)$.

PROPOSITION 2.2. *The inclusion $\mathbb{K} \to T(V)$ descends to an inclusion $\mathbb{K} \to \mathrm{Cl}(V;B)$. The inclusion $V \to T(V)$ descends to an inclusion $V \to \mathrm{Cl}(V;B)$.*

PROOF. Consider the linear map

$$f\colon V \to \mathrm{End}(\wedge(V)),\ v \mapsto \epsilon(v) + \iota(B^\flat(v)).$$

and its extension to an algebra homomorphism $f_T\colon T(V) \to \mathrm{End}(\wedge(V))$. The commutation relations (16) show that $f(v)f(w)+f(w)f(v) = 2B(v,w)1$. Hence f_T vanishes on the ideal $\mathcal{I}(V;B)$, and therefore descends to an algebra homomorphism

$$f_{\mathrm{Cl}}\colon \mathrm{Cl}(V;B) \to \mathrm{End}(\wedge(V)), \tag{17}$$

i.e. $f_{\mathrm{Cl}} \circ \pi = f_T$ where $\pi\colon T(V) \to \mathrm{Cl}(V;B)$ is the projection. Since $f_T(1) = 1$, we see that $\pi(1) \neq 0$, i.e. the inclusion $\mathbb{K} \hookrightarrow T(V)$ descends to an inclusion $\mathbb{K} \hookrightarrow \mathrm{Cl}(V;B)$. Similarly, from $f_T(v).1 = v$ we see that the inclusion $V \hookrightarrow T(V)$ descends to an inclusion $V \hookrightarrow \mathrm{Cl}(V;B)$. $\square$

The Proposition shows that V is a subspace of $\mathrm{Cl}(V;B)$. We may thus characterize $\mathrm{Cl}(V;B)$ as the unital associative algebra, with generators $v \in V$ and relations

$$vw + wv = 2B(v,w), \quad v,w \in V. \tag{18}$$

Let us view $T(V) = \bigoplus_k T^k(V)$ as a filtered super algebra (cf. Appendix A), with the $\mathbb{Z}_2$-grading and filtration inherited from the $\mathbb{Z}$-grading. Since the elements $v \otimes w + w \otimes v - 2B(v,w)1$ are even, of filtration degree 2, the ideal $\mathcal{I}(V;B)$ is a filtered super subspace of $T(V)$, and hence $\mathrm{Cl}(V;B)$ inherits the structure of a filtered super algebra. Simply put, the $\mathbb{Z}_2$-grading and filtration on $\mathrm{Cl}(V;B)$ are defined by the condition that the generators $v \in V$ are odd, of filtration degree 1. In the decomposition

$$\mathrm{Cl}(V;B) = \mathrm{Cl}^{\bar{0}}(V;B) \oplus \mathrm{Cl}^{\bar{1}}(V;B)$$

the two summands are spanned by products $v_1 \cdots v_k$ with k even, respectively odd. From now on, we will always regard $\mathrm{Cl}(V;B)$ as a filtered super algebra (unless stated otherwise), in particular commutators $[\cdot,\cdot]$ will be in the $\mathbb{Z}_2$-graded sense. In this notation, the defining relations for the Clifford algebra become

$$[v,w]=2B(v,w),\quad v,w\in V.$$

If $\dim V=n$, and e_i are an orthogonal basis of V, then (using the same notation as for the exterior algebra), the products

$$e_I=e_{i_1}\cdots e_{i_k},\quad I=\{i_1,\ldots,i_k\}\subset\{1,\ldots,n\},$$

with the convention $e_\emptyset=1$, span $\mathrm{Cl}(V;B)$. We will see in Section 2.5 that the e_I are a basis.

2.2. Universal property, functoriality. The Clifford algebra is characterized by the following universal property:

PROPOSITION 2.3. *Let $\mathcal{A}$ be a filtered super algebra, and $f\colon V\to\mathcal{A}^{(1)}$ a linear map satisfying*

$$f(v_1)f(v_2)+f(v_2)f(v_1)=2B(v_1,v_2)\cdot 1,\quad v_1,v_2\in V.$$

Then f extends uniquely to a morphism of filtered super algebras $\mathrm{Cl}(V;B)\to\mathcal{A}$.

PROOF. By the universal property of the tensor algebra, f extends to an algebra homomorphism $f_{T(V)}\colon T(V)\to\mathcal{A}$. The property $f(v_1)f(v_2)+f(v_2)f(v_1)=2B(v_1,v_2)\cdot 1$ shows that f vanishes on the ideal $\mathcal{I}(V;B)$, and hence descends to the Clifford algebra. Uniqueness is clear, since the Clifford algebra is generated by elements of V. $\square$

REMARK 2.4. We can also view $\mathrm{Cl}(V;B)$ just as super algebra (forgetting the filtration), as a filtered algebra (forgetting the $\mathbb{Z}_2$-grading), or as an (ordinary) algebra, and formulate universal properties for each of these contexts.

Suppose B_1,B_2 are symmetric bilinear forms on V_1,V_2, and $f\colon V_1\to V_2$ is a linear map such that

$$B_2(f(v),f(w))=B_1(v,w),\quad v,w\in V_1.$$

Viewing f as a map into $\mathrm{Cl}(V_2;B_2)$, the universal property provides a unique extension to a morphism of filtered super algebras

$$\mathrm{Cl}(f)\colon\mathrm{Cl}(V_1;B_1)\to\mathrm{Cl}(V_2;B_2).$$

Clearly,

$$\mathrm{Cl}(f_1\circ f_2)=\mathrm{Cl}(f_1)\circ\mathrm{Cl}(f_2),\quad \mathrm{Cl}(\mathrm{id}_V)=\mathrm{id}_{\mathrm{Cl}(V)}\,.$$

The functoriality gives in particular a group homomorphism

$$\mathrm{O}(V;B)\to\mathrm{Aut}(\mathrm{Cl}(V;B)),\ g\mapsto\mathrm{Cl}(g)$$

into automorphisms of $\mathrm{Cl}(V;B)$ (preserving $\mathbb{Z}_2$-grading and filtration). We will usually just write g in place of $\mathrm{Cl}(g)$. For example, the involution $v\mapsto -v$ lies in $\mathrm{O}(V;B)$, hence it defines an involutive algebra automorphism

Π of $\mathrm{Cl}(V;B)$ called the *parity automorphism*. The ± 1 eigenspaces are the even and odd part of the Clifford algebra, respectively.

Suppose again that (V_1, B_1) and (V_2, B_2) are two vector spaces with symmetric bilinear forms, and consider the direct sum $(V_1 \oplus V_2,\ B_1 \oplus B_2)$. Then

$$\mathrm{Cl}(V_1 \oplus V_2; B_1 \oplus B_2) = \mathrm{Cl}(V_1; B_1) \otimes \mathrm{Cl}(V_2; B_2)$$

as filtered super algebras. This follows since $\mathrm{Cl}(V_1; B_1) \otimes \mathrm{Cl}(V_2; B_2)$ satisfies the universal property of the Clifford algebra over $(V_1 \oplus V_2; B_1 \oplus B_2)$. In particular, if $\mathrm{Cl}(n, m)$ denotes the Clifford algebra for $\mathbb{K}^{n,m}$ we have

$$\mathrm{Cl}(n,m) = \mathrm{Cl}(1,0) \otimes \cdots \otimes \mathrm{Cl}(1,0) \otimes \mathrm{Cl}(0,1) \otimes \cdots \otimes \mathrm{Cl}(0,1),$$

(using the $\mathbb{Z}_2$-graded tensor product).

2.3. The Clifford algebras $\mathrm{Cl}(n,m)$. Consider the case $\mathbb{K} = \mathbb{R}$. For n, m small one can determine the algebras $\mathrm{Cl}(n,m) = \mathrm{Cl}(\mathbb{R}^{n,m})$ by hand.

Proposition 2.5. *For $\mathbb{K} = \mathbb{R}$, one has the following isomorphisms of the Clifford algebras $\mathrm{Cl}(n,m)$ with $n + m \leq 2$:*

$$\mathrm{Cl}(0,1) \cong \mathbb{C}, \qquad \Pi(z) = \overline{z},$$

$$\mathrm{Cl}(1,0) \cong \mathbb{R} \oplus \mathbb{R}, \qquad \Pi(u,v) = (v,u),$$

$$\mathrm{Cl}(0,2) \cong \mathbb{H}, \qquad \Pi = \mathrm{Ad}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad i.e. \quad \Pi(A) = \overline{A},$$

$$\mathrm{Cl}(1,1) \cong \mathrm{Mat}_2(\mathbb{R}), \qquad \Pi = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathrm{Cl}(2,0) \cong \mathrm{Mat}_2(\mathbb{R}), \qquad \Pi = \mathrm{Ad}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Here $\mathbb{C}$ and $\mathbb{H}$ are viewed as algebras over $\mathbb{R}$, and $\mathrm{Mat}_2(\mathbb{R}) = \mathrm{End}(\mathbb{R}^2)$ is the algebra of real 2×2-matrices.

Proof. By the universal property, an algebra $\mathcal{A}$ of dimension 2^{n+m} is isomorphic to $\mathrm{Cl}(n,m)$ if there exists a linear map $f\colon \mathbb{R}^{n,m} \to \mathcal{A}$ satisfying $f(e_i)f(e_j) + f(e_j)f(e_i) = \pm 2\delta_{ij}$, with a plus sign for $i \leq n$ and a minus sign for $i > n$. We write down these maps for $n + m \leq 2$:

$\mathrm{Cl}(0,1)$

$$f(e_1) = \sqrt{-1}.$$

$\mathrm{Cl}(1,0)$

$$f(e_1) = (1, -1).$$

$\mathrm{Cl}(0,2)$

$$f(e_1) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

$\mathrm{Cl}(1,1)$

$$f(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Cl(2, 0)

$$f(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One easily checks that the indicated automorphism Π is $+1$ on the even part and -1 on the odd part. (For $n + m = 2$ the parity automorphism is given as conjugation by $f(e_1)f(e_2)$, see 2.7 below.) □

The full classification of the Clifford algebras $\mathrm{Cl}(n, m)$ may be found in the book by Lawson-Michelsohn [**51**] or in the monograph by Budinich-Trautman [**16**]. The Clifford algebras $\mathrm{Cl}(n, m)$ exhibit a remarkable mod 8 periodicity. For any algebra $\mathcal{A}$ over $\mathbb{K}$ (her $\mathbb{K} = \mathbb{R}$), let $\mathrm{Mat}_k(\mathcal{A}) = \mathcal{A} \otimes \mathrm{Mat}_k(\mathbb{K})$ be the algebra of $k \times k$ matrices with entries in $\mathcal{A}$. Then

$$\mathrm{Cl}(n + 8, m) \cong \mathrm{Mat}_{16}(\mathrm{Cl}(n, m)) \cong \mathrm{Cl}(n, m + 8).$$

These isomorphisms are related to the mod 8 periodicity in real K-theory [**8**].

2.4. The Clifford algebras $\mathbb{C}l(n)$. For $\mathbb{K} = \mathbb{C}$ the pattern is simpler. Denote by $\mathbb{C}l(n)$ the Clifford algebra of $\mathbb{C}^n$.

PROPOSITION 2.6. *One has the following isomorphisms of algebras over* $\mathbb{C}$,

$$\mathbb{C}l(2m) = \mathrm{Mat}_{2^m}(\mathbb{C}), \quad \mathbb{C}l(2m + 1) = \mathrm{Mat}_{2^m}(\mathbb{C}) \oplus \mathrm{Mat}_{2^m}(\mathbb{C}).$$

More precisely, $\mathbb{C}l(2m) = \mathrm{End}(\wedge\mathbb{C}^m)$ *as a super algebra, while* $\mathbb{C}l(2m+1) = \mathrm{End}(\wedge\mathbb{C}^m) \otimes (\mathbb{C} \oplus \mathbb{C})$ *as a super algebra, where the parity automorphism of* $\mathbb{C} \oplus \mathbb{C}$ *is* $(u, v) \mapsto (v, u)$*, and using the tensor product of super algebras.*

PROOF. Consider first the case $n = 2$. The map $f\colon \mathbb{C}^2 \to \mathrm{End}(\mathbb{C}^2)$,

$$f(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$$

extends, by the universality property, to an isomorphism $\mathbb{C}l(2) \to \mathrm{End}(\mathbb{C}^2)$. The resulting $\mathbb{Z}_2$-grading on $\mathrm{End}(\mathbb{C}^2)$ is induced by the $\mathbb{Z}_2$-grading on $\mathbb{C}^2$ where the first component is even and the second is odd. Equivalently, it corresponds to the identification $\mathbb{C}^2 \cong \wedge\mathbb{C}$. This shows $\mathbb{C}l(2) \cong \mathrm{End}(\wedge\mathbb{C})$ as super algebras. For $\mathbb{C}^{2m} = \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2$ we hence obtain

$$\begin{aligned} \mathbb{C}l(2m) &= \mathbb{C}l(2) \otimes \cdots \otimes \mathbb{C}l(2) \\ &\cong \mathrm{End}(\wedge\mathbb{C}) \otimes \cdots \otimes \mathrm{End}(\wedge\mathbb{C}) \\ &= \mathrm{End}(\wedge\mathbb{C} \otimes \cdots \otimes \wedge\mathbb{C}) \\ &= \mathrm{End}(\wedge\mathbb{C}^m), \end{aligned}$$

as super algebras (using the tensor product of super algebras).

For $n = 1$ we have $\mathbb{C}l(1) = \mathbb{C} \oplus \mathbb{C}$ with parity automorphism $\Pi(u, v) = (v, u)$, by the same argument as for the real Clifford algebra $\mathrm{Cl}(1, 0)$. Hence

$$\mathbb{C}l(2m + 1) = \mathbb{C}l(2m) \otimes \mathbb{C}l(1) = \mathrm{End}(\wedge\mathbb{C}^m) \otimes (\mathbb{C} \oplus \mathbb{C})$$

as super algebras. □

The mod 2 periodicity

$$\mathbb{C}l(n+2) \cong \mathrm{Mat}_2(\mathbb{C}l(n))$$

apparent in this classification result is related to the mod 2 periodicity in complex K-theory [**8**].

REMARK 2.7. The result shows in particular that there is an isomorphism of (ungraded) algebras,

$$\mathbb{C}l(2m-1) \cong \mathbb{C}l^{\bar{0}}(2m).$$

This can be directly seen as follows: By the universal property, the map $\mathbb{C}^{2m-1} \to \mathbb{C}l^{\bar{0}}(2m)$, $e_i \mapsto \sqrt{-1}\, e_i e_{2m}$ for $i < 2m$ extends to an algebra homomorphism $\mathbb{C}l(2m-1) \to \mathbb{C}l^{\bar{0}}(2m)$.

2.5. Symbol map and quantization map. We now return to the representation

$$f_{\mathrm{Cl}} \colon \mathrm{Cl}(V;B) \to \mathrm{End}(\wedge V), \quad f_{\mathrm{Cl}}(v) = \epsilon(v) + \iota(B^\flat(v))$$

of the Clifford algebra, (see (17)). One defines the *symbol map* by the action on the element $1 \in \wedge(V)$:

$$\sigma \colon \mathrm{Cl}(V;B) \to \wedge(V),\ x \mapsto f_{\mathrm{Cl}}(x).1$$

where $1 \in \wedge^0(V) = \mathbb{K}$.

PROPOSITION 2.8. *The symbol map is an isomorphism of filtered super vector spaces. In low degrees,*

$$\begin{aligned} \sigma(1) &= 1 \\ \sigma(v) &= v \\ \sigma(v_1v_2) &= v_1 \wedge v_2 + B(v_1,v_2), \\ \sigma(v_1v_2v_3) &= v_1 \wedge v_2 \wedge v_3 + B(v_2,v_3)v_1 - B(v_1,v_3)v_2 + B(v_1,v_2)v_3. \end{aligned}$$

PROOF. Let $e_i \in V$ be an orthogonal basis. Since the operators $f(e_i)$ commute (in the grade sense), we find

$$\sigma(e_{i_1} \cdots e_{i_k}) = e_{i_1} \wedge \cdots \wedge e_{i_k},$$

for $i_1 < \cdots < i_k$. This directly shows that the symbol map is an isomorphism: It takes the element $e_I \in \mathrm{Cl}(V;B)$ to the corresponding element $e_I \in \wedge(V)$. The formulas in low degrees are obtained by straightforward calculation. □

The inverse of the symbol map is called the *quantization map*

$$q\colon \wedge(V) \to \mathrm{Cl}(V;B).$$

In terms of the basis, $q(e_I) = e_I$. In low degrees,

$$\begin{aligned} q(1) &= 1, \\ q(v) &= v, \\ q(v_1 \wedge v_2) &= v_1 v_2 - B(v_1, v_2), \\ q(v_1 \wedge v_2 \wedge v_3) &= v_1 v_2 v_3 - B(v_2, v_3) v_1 + B(v_1, v_3) v_2 - B(v_1, v_2) v_3. \end{aligned}$$

PROPOSITION 2.9. *The symbol map induces an isomorphism of graded super algebras,*

$$\mathrm{gr}(\mathrm{Cl}(V)) \to \wedge(V).$$

PROOF. Since the symbol map $\mathrm{Cl}(V) \to \wedge(V)$ is an isomorphism of filtered super spaces, the associated graded map $\mathrm{gr}(\mathrm{Cl}(V)) \to \mathrm{gr}(\wedge(V)) = \wedge(V)$ is an isomorphism of graded super spaces. To check that the induced map preserves products, we must show that the symbol map intertwines products up to lower order terms. That is, for $x \in \mathrm{Cl}(V)^{(k)}$ and $y \in \mathrm{Cl}(V)^{(l)}$ we have $\sigma(xy) - \sigma(x)\sigma(y) \in \wedge^{k+l-1}(V)$. But this is clear from

$$\begin{aligned} \sigma(v_1 \cdots v_r) &= (\epsilon(v_1) + \iota(B^\flat(v_1)) \cdots (\epsilon(v_r) + \iota(B^\flat(v_r)).1 \\ &= v_1 \wedge \cdots \wedge v_k \mod \wedge^{r-1}(V), \end{aligned}$$

for $v_i \in V$. ☐

The quantization map has the following alternative description.

PROPOSITION 2.10. *The quantization map is given by graded symmetrization. That is, for $v_1, \ldots, v_k \in V$,*

$$q(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \mathrm{sign}(s) v_{s(1)} \cdots v_{s(k)}.$$

Here $\mathfrak{S}_k$ is the group of permutations of $1, \ldots, k$ and $\mathrm{sign}(s) = \pm 1$ is the parity of a permutation s.

PROOF. By linearity, it suffices to check for the case that the v_j are elements of an orthonormal basis $e_1, \ldots, e_n$ of V, that is $v_j = e_{i_j}$ (the indices i_j need not be ordered or distinct). If the i_j are all distinct, then the e_{i_j} Clifford commute in the graded sense, and the right hand side equals $e_{i_1} \cdots e_{i_k} \in \mathrm{Cl}(V; B)$, which coincides with the left hand side. If any two e_{i_j} coincide, then both sides are zero. ☐

2.6. Transposition. An *anti-automorphism* of an algebra $\mathcal{A}$ is an invertible linear map $f \colon \mathcal{A} \to \mathcal{A}$ with the property $f(ab) = f(b)f(a)$ for all $a, b \in \mathcal{A}$. Put differently, if $\mathcal{A}^{\mathrm{op}}$ is $\mathcal{A}$ with the opposite algebra structure $a \cdot_{\mathrm{op}} b := ba$, an anti-automorphism is an algebra isomorphism $\mathcal{A} \to \mathcal{A}^{\mathrm{op}}$.

The tensor algebra carries a unique involutive anti-automorphism that is equal to the identity on $V \subset T(V)$. It is called the *canonical anti-automorphism* or *transposition*, and is given by

$$(v_1 \otimes \cdots \otimes v_k)^\top = v_k \otimes \cdots \otimes v_1.$$

Since transposition preserves the ideal $\mathcal{I}(V)$ defining the exterior algebra $\wedge(V)$, it descends to an anti-automorphism of the exterior algebra, $\phi \mapsto \phi^\top$. In fact, since transposition is given by a permutation of length $(k-1)+\cdots+2+1 = k(k-1)/2$, we have

$$\phi^\top = (-1)^{k(k-1)/2}\phi, \quad \phi \in \wedge^k(V). \tag{19}$$

Given a symmetric bilinear form B on V, the canonical anti-automorphism of the tensor algebra also preserves the ideal $\mathcal{I}(V;B)$. Hence it descends to an anti-automorphism of $\mathrm{Cl}(V;B)$, still called the *canonical anti-automorphism* or *transposition*, with

$$(v_1\cdots v_k)^\top = v_k \cdots v_1.$$

The quantization map $q\colon \wedge(V) \to \mathrm{Cl}(V;B)$ intertwines the transposition maps for $\wedge(V)$ and $\mathrm{Cl}(V;B)$. This is sometimes useful for computations.

EXAMPLE 2.11. Suppose $\phi \in \wedge^k(V)$, and consider the square of $q(\phi)$. The element $q(\phi)^2 \in \mathrm{Cl}(V)$ is even, and is hence contained in $\mathrm{Cl}^{\bar{0}}_{(2k)}(V)$. But $(q(\phi)^2)^\top = (q(\phi)^\top)^2 = q(\phi)^2$ since $q(\phi)^\top = q(\phi^\top) = \pm q(\phi)$. It follows that

$$q(\phi)^2 \in q\big(\wedge^0(V) \oplus \wedge^4(V) \oplus \cdots \oplus \wedge^{4r}(V)\big),$$

where r is the largest number with $2r \le k$.

2.7. Chirality element. Let $\dim V = n$. Then any generator $\Gamma_\wedge \in \det(V) := \wedge^n(V)$ quantizes to give an element $\Gamma = q(\Gamma_\wedge)$. This element (or suitable normalizations of this element) is called the *chirality element* of the Clifford algebra. The square Γ^2 of the chirality element is always a scalar, as is immediate by choosing an orthogonal basis e_i, and letting $\Gamma = e_1\cdots e_n$. In fact, since $\Gamma^\top = (-1)^{n(n-1)/2}\Gamma$ by (19), we have

$$\Gamma^2 = (-1)^{n(n-1)/2}\prod_{i=1}^n B(e_i,e_i).$$

In the case $\mathbb{K} = \mathbb{C}$ and $V = \mathbb{C}^n$ we can always normalize Γ to satisfy $\Gamma^2 = 1$; this normalization determines Γ up to sign. Another important case where Γ admits such a normalization is that of a vector space V with split bilinear form. Choose a pair of transverse Lagrangian subspaces to identify $V = F^* \oplus F$, and pick dual bases $e_1,\ldots,e_m$ of F and $f^1,\ldots,f^m$ of F^*. Then $B(e_i,f^j) = \frac{1}{2}\delta_i^j$, and the vectors $e_i - f^i,\ e_i + f^i,\ i = 1,\ldots,m$ form an orthogonal basis of V. Using $(e_i - f^i)^2 = -1, (e_i+f^i)^2 = 1$ we see that

$$\Gamma = (e_1 - f^1)(e_1 + f^1)\cdots(e_m - f^m)(e_m + f^m) \tag{20}$$

satisfies $\Gamma^2 = 1$. Returning to the general case, we observe that $\Gamma v = (-1)^{n-1} v\Gamma$ for all $v \in V$, e.g. by checking in an orthogonal basis. (If $v = e_i$, then v anti-commutes with all e_j for $j \ne i$ in the product $\Gamma = e_1\cdots e_n$, and commutes with e_i. Hence we obtain $n-1$ sign changes.)

$$\Gamma v = \begin{cases} v\Gamma & \text{if } n \text{ is odd} \\ -v\Gamma & \text{if } n \text{ is even} \end{cases}$$

Thus, if n is odd then Γ lies in the center of $\mathrm{Cl}(V;B)$, *viewed as an ordinary algebra*. If n is even, the element Γ is even, and lies in the center of $\mathrm{Cl}^{\bar{0}}(V;B)$. Furthermore, in this case

$$\Pi(x) = \Gamma x \Gamma^{-1},$$

for all $x \in \mathrm{Cl}(V;B)$, i.e. the chirality element *implements* the parity automorphism.

2.8. The trace and the super-trace. For any super algebra $\mathcal{A}$ and (super) vector space Y, a *Y-valued trace* on $\mathcal{A}$ is an even linear map $\mathrm{tr}_s\colon \mathcal{A} \to Y$ vanishing on the subspace $[\mathcal{A},\mathcal{A}]$ spanned by super-commutators: That is, $\mathrm{tr}_s([x,y]) = 0$ for $x, y \in \mathcal{A}$.

PROPOSITION 2.12. *Let $n = \dim V$. The linear map*

$$\mathrm{tr}_s\colon \mathrm{Cl}(V;B) \to \det(V)$$

given as the quotient map to $\mathrm{Cl}_{(n)}(V;B)/\mathrm{Cl}_{(n-1)}(V;B) \cong \wedge^n(V) = \det(V)$, *is* $\det(V)$*-valued trace on the super algebra* $\mathrm{Cl}(V;B)$.

PROOF. Let e_i be an orthogonal basis, and e_I the associated basis of $\mathrm{Cl}(V;B)$. Then $\mathrm{tr}_s(e_I) = 0$ unless $I = \{1,\ldots,n\}$. The product e_I, e_J is of the form $e_I e_J = c e_K$ where $K = (I \cup J) - (I \cap J)$ and $c \in \mathbb{K}$. Hence $\mathrm{tr}_s(e_I e_J) = 0 = \mathrm{tr}_s(e_J e_I)$ unless $I \cap J = \emptyset$ and $I \cup J = \{1,\ldots,n\}$. But in case $I \cap J = \emptyset$, e_I, e_J super-commute: $[e_I, e_J] = 0$. $\square$

The Clifford algebra also carries an *ordinary trace*, vanishing on ordinary commutators.

PROPOSITION 2.13. *The formula*

$$\mathrm{tr}\colon \mathrm{Cl}(V;B) \to \mathbb{K}, \quad x \mapsto \sigma(x)_{[0]}$$

defines an (ordinary) trace on $\mathrm{Cl}(V;B)$, *that is* $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ *for all* $x, y \in \mathrm{Cl}(V;B)$. *The trace satisfies* $\mathrm{tr}(x^\top) = \mathrm{tr}(x)$ *and* $\mathrm{tr}(1) = 1$, *and is related to the super-trace by the formula,*

$$\mathrm{tr}_s(\Gamma x) = \mathrm{tr}(x)\,\Gamma_\wedge$$

where $\Gamma = q(\Gamma_\wedge)$ *is the chirality element in the Clifford algebra defined by a choice of generator of* $\det(V)$.

PROOF. Again, we use an orthogonal basis e_i of V. The definition gives $\mathrm{tr}(e_\emptyset) = 1$, while $\mathrm{tr}(e_I) = 0$ for $I \neq \emptyset$. We will show $\mathrm{tr}(e_I e_J) = \mathrm{tr}(e_J e_I)$. We have $e_I e_J = c e_K$ where $K = (I \cup J) - (I \cap J)$ and $c \in \mathbb{K}$. If $I \neq J$ the set K is non-empty, hence $\mathrm{tr}(e_I e_J) = 0 = \mathrm{tr}(e_J e_I)$. If $I = J$ the trace property is trivial. To check the formula relating trace and super-trace we may assume $\Gamma_\wedge = e_1 \cdots e_n$. For $x = e_J$ we see that $\mathrm{tr}_s(\Gamma x)$ vanishes unless $J = \emptyset$, in which case we obtain $\mathrm{tr}_s(\Gamma) = \Gamma_\wedge$. $\square$

2.9. Extension of the bilinear form. The symmetric bilinear form on V extends to a symmetric bilinear form on the exterior algebra $\wedge(V)$, by setting $B(\phi,\psi)=0$ for $|\phi|\neq|\psi|$ and

$$B(v_1\wedge\cdots\wedge v_k,\ w_1\wedge\cdots\wedge w_k)=\det\big(B(v_i,w_j)_{i,j}\big).$$

On the other hand, using the trace on $\mathrm{Cl}(V;B)$ we also have an extension to the Clifford algebra:

$$B(x,y)=\mathrm{tr}(x^\top y),\quad x,y\in\mathrm{Cl}(V;B).$$

PROPOSITION 2.14. *The quantization map q intertwines the bilinear forms on $\wedge(V)$ and $\mathrm{Cl}(V;B)$.*

PROOF. We check in an orthogonal basis e_i of V. Indeed, for $I\neq J$ $B(e_I,e_J)$ vanishes in $\wedge(V)$, but also in $\mathrm{Cl}(V;B)$ since $e_I^\top e_J=\pm e_Ie_J$ has trace zero. On the other hand, taking $I=J=\{i_1,\ldots,i_k\}$ we get $B(e_I,e_I)=\prod_{j=1}^k B(e_{i_j},e_{i_j})$ in both the Clifford and exterior algebras. □

2.10. Lie derivatives and contractions. For any vector space V, there is an isomorphism of graded super vector spaces

$$\mathrm{Der}(T(V))\cong\mathrm{Hom}(V,T(V))=T(V)\otimes V^*,$$

where $V\cong T(V)^1$ is regarded as a graded super vector space concentrated in degree 1, and $\mathrm{Der}(T(V))$ are the derivations of $T(V)$ as a graded super algebra. Indeed, any derivation of $T(V)$ is uniquely determined by its restriction to V; conversely any linear map $D\colon V\to T(V)^{1+r}$ of degree $r\in\mathbb{Z}$ extends to an element of $\mathrm{Der}(T(V))^r$ by the derivation property,

$$D(v_1\otimes\cdots\otimes v_k)=\sum_{i=1}^k(-1)^{r(i-1)}v_1\otimes\cdots\otimes Dv_i\otimes\cdots\otimes v_k,$$

for $v_1,\ldots,v_k\in V$. (For a detailed proof, see Chevalley [**22**, page 27].)

The graded super Lie algebra $\mathrm{Der}(T(V))$ has non-vanishing components in degrees $r\geq -1$. The component $\mathrm{Der}(T(V))^{-1}$ is the space $\mathrm{Hom}(V,\mathbb{K})=V^*$, acting by *contractions* $\iota(\alpha),\ \alpha\in V^*$:

$$\iota(\alpha)(v_1\otimes\cdots\otimes v_k)=\sum_{i=1}^k(-1)^{i-1}\langle\alpha,v_i\rangle\ v_1\otimes\cdots\widehat{v_i}\cdots\otimes v_k.$$

The component $\mathrm{Der}(T(V))^0=\mathrm{Hom}(V,V)=\mathfrak{gl}(V)$ is the space of *Lie derivatives* $L_A\in\mathrm{Der}(T(V)),\quad A\in\mathfrak{gl}(V)$:

$$L_A(v_1\otimes\cdots\otimes v_k)=\sum_{i=1}^k v_1\otimes\cdots\otimes L_A(v_i)\otimes\cdots\otimes v_k.$$

We have $\mathrm{Der}(T(V))^{-1}\oplus\mathrm{Der}(T(V))^0\ \cong\ V^*\rtimes\mathfrak{gl}(V)$ as graded super Lie algebras.

Remark 2.15. A parallel discussion describes the derivations of $T(V)$ as an ordinary graded algebra; here one omits the sign $(-1)^{r(i-1)}$ in the formula for the extension of D.

Both contractions and Lie derivatives preserve the ideal $\mathcal{I}(V)$ defining the exterior algebra, and hence descend to derivations of $\wedge(V)$, still called contractions and Lie derivatives. This defines a morphism of graded super Lie algebras $V^* \rtimes \mathfrak{gl}(V) \to \mathrm{Der}(\wedge(V))$.

Given a symmetric bilinear form B on V, the contraction operators also preserve the ideal $\mathcal{I}(V; B)$ since

$$\iota(\alpha)(v_1 \otimes v_2 + v_2 \otimes v_1 - 2B(v_1, v_2)) = 0, \quad v_1, v_2 \in V.$$

Hence they descend to odd derivations $\iota(\alpha)$ of $\mathrm{Cl}(V; B)$ of filtration degree -1, given as

$$\iota(\alpha)(v_1 \cdots v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \alpha, v_i \rangle v_1 \cdots \widehat{v_i} \cdots v_k. \tag{21}$$

On the other hand, the Lie derivatives L_A on $T(V)$ preserve the ideal $\mathcal{I}(V; B)$ if and only if $A \in \mathfrak{o}(V; B)$, that is $B(Av_1, v_2) + B(v_1, Av_2) = 0$ for all v_1, v_2. Under this condition, L_A descends to an even derivation of filtration degree 0

$$L_A(v_1 \cdots v_k) = \sum_{i=1}^{k} v_1 \cdots L_A(v_i) \cdots v_k$$

on the Clifford algebra. Together with the contractions, this gives a morphism of filtered super Lie algebras $V^* \rtimes \mathfrak{o}(V; B) \to \mathrm{Der}(\mathrm{Cl}(V; B))$:

$$[\iota(\alpha_1), \iota(\alpha_2)] = 0,\ [L_{A_1}, L_{A_2}] = L_{[A_1, A_2]},\ [L_A, \iota(\alpha)] = \iota(A.\alpha),$$

where $A.\alpha = -A^*\alpha$ with A^* the dual map.

Proposition 2.16. *The symbol map intertwines the action of $V^* \rtimes \mathfrak{o}(V; B)$ by contractions and Lie derivatives on $\mathrm{Cl}(V; B)$ with the corresponding action on $\wedge(V)$.*

Proof. It suffices to check on elements $\phi = v_1 \wedge \cdots \wedge v_k \in \wedge(V)$ where $v_1, \ldots, v_k$ are pairwise orthogonal. Then $q(\phi) = v_1 \cdots v_k$, and the quantization of $\iota(\alpha)\phi$ (given by (15)) coincides with $\iota(\alpha)(q(\phi))$ (given by (21)). The argument for the Lie derivatives is similar. □

Any element $v \in V$ defines a derivation of $\mathrm{Cl}(V; B)$ by super commutator: $x \mapsto [v, x]$. For generators $w \in V$, we have $[v, w] = 2B(v, w) = 2\langle B^\flat(v),\ w \rangle$. This shows that this derivation agrees with the contraction by $2B^\flat(v)$:

$$[v, \cdot] = 2\iota(B^\flat(v)) \tag{22}$$

As a simple application, we find:

LEMMA 2.17. *The center of the filtered super algebra* $\mathrm{Cl}(V;B)$ *is the exterior algebra over* $\mathrm{rad}(B) = \ker B^\flat$. *Hence, if* B *is non-degenerate the center consists of the scalars.*

We stress that the Lemma refers to the 'super-center', not the center of $\mathrm{Cl}(V;B)$ as an ordinary algebra.

PROOF. Indeed, suppose x lies in the center. Then $0 = [v, x] = 2\iota(B^\flat(v))x$ for all $v \in V$. Hence $\sigma(x)$ is annihilated by all contractions $B^\flat(v))$, and is therefore an element of the exterior algebra over $\mathrm{ann}(\mathrm{ran}(B^\flat)) = \ker(B^\flat)$. Consequently $x = q(\sigma(x))$ is in $\mathrm{Cl}(\ker(B^\flat)) = \wedge(\ker(B^\flat))$. □

2.11. The Lie algebra $q(\wedge^2(V))$**.** The following important fact relates the 'quadratic elements' of the Clifford algebra to the Lie algebra $\mathfrak{o}(V;B)$.

THEOREM 2.18. *The elements* $q(\lambda),\ \lambda \in \wedge^2(V)$ *span a Lie subalgebra of* $\mathrm{Cl}(V;B)$. *Let* $\{\cdot,\cdot\}$ *be the induced Lie bracket on* $\wedge^2(V)$ *so that*

$$[q(\lambda), q(\lambda')] = q(\{\lambda, \lambda'\}).$$

The transformation $v \mapsto A_\lambda(v) = [q(\lambda), v]$ *defines an element* $A_\lambda \in \mathfrak{o}(V;B)$, *and the map*

$$\wedge^2(V) \to \mathfrak{o}(V;B), \quad \lambda \mapsto A_\lambda$$

is a Lie algebra homomorphism. One has $L_{A_\lambda} = [q(\lambda), \cdot]$ *as derivations of* $\mathrm{Cl}(V;B)$.

PROOF. By definition, $A_\lambda(v) = [q(\lambda), v] = -2\iota(B^\flat(v))q(\lambda)$. Hence

$$A_\lambda(v) = -2\iota(B^\flat(v))\lambda$$

since the quantization map intertwines the contractions of the exterior and Clifford algebras. We have $A_\lambda \in \mathfrak{o}(V;B)$ since

$$B(A_\lambda(v), w) = -2\iota(B^\flat(w))A_\lambda(v) = -2\iota(B^\flat(w))\iota(B^\flat(v))\lambda$$

is anti-symmetric in v, w. It follows that $L_{A_\lambda} = [q(\lambda), \cdot]$ since the two sides are derivations which agree on generators. Define a bracket $\{\cdot,\cdot\}$ on $\wedge^2(V)$ by

$$\{\lambda, \lambda'\} = L_{A_\lambda}\lambda' \tag{23}$$

(using the Lie derivatives on $\wedge(V)$). The calculation

$$[q(\lambda), q(\lambda')] = L_{A_\lambda}q(\lambda') = q(L_{A_\lambda}\lambda') = q(\{\lambda, \lambda'\})$$

shows that q intertwines $\{\cdot,\cdot\}$ with the Clifford commutator; in particular $\{\cdot,\cdot\}$ is a Lie bracket. Furthermore, from

$$[q(\lambda), [q(\lambda'), v]] - [q(\lambda'), [q(\lambda), v]] = [[q(\lambda), q(\lambda')], v] = [q(\{\lambda, \lambda'\}), v]$$

we see that $[A_\lambda, A_{\lambda'}] = A_{\{\lambda,\lambda'\}}$, that is, the map $\lambda \mapsto A_\lambda$ is a Lie algebra homomorphism. □

COROLLARY 2.19. *Relative to the bracket* $\{\cdot,\cdot\}$ *on* $\wedge^2(V)$*, the map*

$$V \rtimes \wedge^2(V) \to V^* \rtimes \mathfrak{o}(V;B), \quad (v,\lambda) \mapsto (B^\flat(v), A_\lambda)$$

is a homomorphism of graded super Lie algebras. We have a commutative diagram,

$$\begin{tikzcd} V \rtimes \wedge^2(V) \arrow[r] \arrow[d, "q"] & V^* \rtimes \mathfrak{o}(V;B) \arrow[d] \\ \mathrm{Cl}(V;B) \arrow[r, "\mathrm{ad}"'] & \mathrm{Der}(\mathrm{Cl}(V;B)) \end{tikzcd}$$

Note that we can think of $V \rtimes \wedge^2(V)$ as a graded super subspace of $\wedge(V)[2]$, using the standard grading on $\wedge(V)$ shifted down by 2. We will see in the following Section 3 that the graded Lie bracket on $V \rtimes \wedge^2(V)$ extends to a graded Lie bracket on all of $\wedge(V)[2]$.

PROPOSITION 2.20. *If* B *is non-degenerate, then the map* $\lambda \mapsto A_\lambda$ *is an isomorphism* $\wedge^2(V) \to \mathfrak{o}(V;B)$*.*

PROOF. In a basis e_i of V, with B-dual basis e^i the inverse map $\mathfrak{o}(V;B) \to \wedge^2(V)$ is given by $A \mapsto \frac{1}{4}\sum_i A(e_i) \wedge e^i$. Indeed, since $B(A(e_i), e^i) = 0$ the quantization of such an element is just $\frac{1}{4}\sum_i A(e_i)e^i \in \mathrm{Cl}(V;B)$, and one directly checks that

$$[\frac{1}{4}\sum_i A(e_i)e^i, v] = A(v)$$

as required. $\square$

The inverse map will be denoted

$$\lambda\colon \mathfrak{o}(V;B) \to \wedge^2(V), \tag{24}$$

and its quantization

$$\gamma = q \circ \lambda\colon \mathfrak{o}(V;B) \to \mathrm{Cl}(V), \tag{25}$$

In a basis e_i if V, with B-dual basis e^i we have

$$\lambda(A) = \frac{1}{4}\sum_i A(e_i) \wedge e^i, \tag{26}$$

hence $\gamma(A) = \frac{1}{4}\sum_i A(e_i)e^i$.

2.12. A formula for the Clifford product. It is sometimes useful to express the Clifford multiplication

$$m_{\mathrm{Cl}}\colon \mathrm{Cl}(V \oplus V) = \mathrm{Cl}(V) \otimes \mathrm{Cl}(V) \to \mathrm{Cl}(V)$$

in terms of the exterior algebra multiplication,

$$m_\wedge\colon \wedge(V \oplus V) = \wedge(V) \otimes \wedge(V) \to \wedge(V).$$

We assume that $n = \dim V < \infty$. Let $e_i \in V,\ i = 1, \ldots, n$ be an orthogonal basis, $e^i \in V^*$ the dual basis, and $e_I \in \wedge(V),\ e^I \in \wedge(V^*)$ the corresponding dual bases indexed by subsets $I \subset \{1, \ldots, n\}$. Then the element

$$\Psi = \sum_I e^I \otimes B^\flat(e_I)^\top \in \wedge(V^*) \otimes \wedge(V^*)$$

is independent of the choice of bases.

Proposition 2.21. *Under the quantization map, the exterior algebra product and the Clifford product are related as follows:*

$$m_{\mathrm{Cl}} \circ (q \otimes q) = q \circ m_\wedge \circ \iota(\Psi)$$

Proof. Let V_i be the 1-dimensional subspace spanned by e_i. Then $\wedge(V)$ is the graded tensor product over all $\wedge(V_i)$, and similarly $\mathrm{Cl}(V)$ is the graded tensor product over all $\mathrm{Cl}(V_i)$. The formula for Ψ factorizes as

$$\Psi = \prod_{i=1}^n \big(1 - e^i \otimes B^\flat(e_i)\big). \tag{27}$$

It hence suffices to prove the formula for the case $V = V_1$. The contraction operator $\iota\big(1 - e^1 \otimes B^\flat(e_1)\big)$ takes the basis elements $1 \otimes 1,\ 1 \otimes e_1,\ e_1 \otimes 1, e_1 \otimes e_1$ to $1 \otimes 1,\ 1 \otimes e_1,\ e_1 \otimes 1, e_1 \otimes e_1 + B(e_1, e_1)$ respectively. Hence $q \circ m_\wedge \circ \iota\big(1 - e^1 \otimes B^\flat(e_1)\big)$ takes these basis elements to $1,\ e_1,\ e_1, B(e_1, e_1)$. But this is the same as the image of the basis elements under Clifford multiplication. $\square$

If $\mathrm{char}(\mathbb{K}) = 0$, we may also write the element Ψ as an exponential:

$$\Psi = \exp\big(-\sum_i e^i \otimes B^\flat(e_i)\big).$$

This follows by rewriting (27) as $\prod_i \exp\big(-e^i \otimes B^\flat(e_i)\big)$, and then writing the product of exponentials as an exponential of a sum.

3. The Clifford algebra as a quantization of the exterior algebra

Using the quantization map, the Clifford algebra $\mathrm{Cl}(V; B)$ may be thought of as $\wedge(V)$ with a new associative product. In this Section make more precise in which sense the Clifford algebra is a quantization of the exterior algebra. Much of the material in this section is motivated by the paper [**48**] of Kostant and Sternberg.

3.1. Differential operators. The prototype of the notion of quantization to be considered here is the algebra of differential operators on a manifold M. For all $k \geq 0$, we let $\mathfrak{D}^{(k)}(M) \subset \mathrm{End}(C^\infty(M))$ denote the space of differential operators of degree $\leq k$. Thus $\mathfrak{D}^{(0)}(M)$ is the algebra of real-valued functions $C^\infty(M)$ (acting by multiplication), and inductively for $k > 0$,

$$\mathfrak{D}^{(k)}(M) = \{D \in \mathrm{End}(C^\infty(M)) |\ \forall f \in C^\infty(M) \colon [D, f] \in \mathfrak{D}^{(k-1)}(M)\}.$$

One may show that this is indeed the familiar notion of differential operators: in local coordinates $q_1, \ldots, q_n$ on M, any $D \in \mathfrak{D}^{(k)}(M)$ has the form

$$D = \sum_{|I| \leq k} a_I(x) (\frac{\partial}{\partial x})^I,$$

using multi-index notation $I = (i_1, \ldots, i_n)$ with

$$|I| = \sum_{j=1}^{n} i_j, \qquad (\frac{\partial}{\partial x})^I = \prod_{j=1}^{n} (\frac{\partial}{\partial x_j})^{i_j}.$$

The composition of operators on $C^\infty(M)$ defines a product

$$\mathfrak{D}^{(k)}(M) \times \mathfrak{D}^{(l)}(M) \to \mathfrak{D}^{(k+l)}(M),$$

making

$$\mathfrak{D}(M) = \bigcup_{k=0}^{\infty} \mathfrak{D}^{(k)}(M)$$

into a filtered algebra. Now let T^*M be the cotangen bundle of M (dual of the tangent bundle), and let $\mathrm{Pol}^k(T^*M) \subset C^\infty(T^*M)$ be the functions whose restriction to each fiber is a polynomial of degree k. Note that $\mathrm{Pol}^k(T^*M)$ is isomorphic to the sections of $S^k(TM)$, the k-th symmetric power of the tangent bundle.

Proposition 3.1. *For every degree k differential operator $D \in \mathfrak{D}^{(k)}(M)$, there is a unique function $\sigma^k(D) \in \mathrm{Pol}^k(T^*M)$ such that for all functions f,*

$$\sigma^k(D) \circ df = \underbrace{[[\cdots [D, f], f] \cdots, f]}_{k \quad \text{times}}$$

Sketch of proof. Writing D in local coordinates as above, the right hand side is the function

$$\sum_{|I|=k} a_I(x) (\frac{\partial f}{\partial x_1})^{i_1} \cdots (\frac{\partial f}{\partial x_k})^{i_k}. \tag{28}$$

In particular, its value at any $x \in M$ depends only the differential $\mathrm{d}_x f$, and is a polynomial of degree k in $\mathrm{d}_x f$. □

The function

$$\sigma^k(D) \in \mathrm{Pol}^k(T^*M) \cong \Gamma^\infty(M, S^k(TM))$$

is called the degree k *principal symbol* of P. By (28), it is given in local coordinates as follows:

$$\sigma^k(D)(x, p) = \sum_{|I|=k} a_I(x) p^I.$$

We see in particular that $\sigma^k(D) = 0$ if and only if $D \in \mathfrak{D}^{(k-1)}(M)$.

We obtain an exact sequence,

$$0 \to \mathfrak{D}^{(k-1)}(M) \to \mathfrak{D}^{(k)}(M) \xrightarrow{\sigma^k} \Gamma^\infty(M; S^k(TM)) \to 0.$$

If D_1, D_2 are differential operators of degrees k_1, k_2, then $D_1 \circ D_2$ is a differential operator of degree $k_1 + k_2$ and

$$\sigma^{k_1+k_2}(D_1 \circ D_2) = \sigma^{k_1}(D_1)\sigma^{k_2}(D_2).$$

The symbol map descends to a morphism of graded algebras,

$$\sigma^\bullet \colon \operatorname{gr}^\bullet \mathfrak{D}(M) \to \Gamma^\infty(M, S^\bullet(TM)).$$

Using a partition of unity, it is not hard to see that this map is an isomorphism.

If D_1, D_2 have degree k_1, k_2, then the degree $k_1 + k_2$ principal symbol of the commutator $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is zero. Hence $[D_1, D_2]$ has degree $k_1 + k_2 - 1$. A calculation of the leading terms shows

$$\sigma_{k_1+k_2-1}([D_1, D_2]) = \{\sigma_{k_1}(D_1),\ \sigma_{k_2}(D_2)\}$$

where $\{\cdot, \cdot\}$ is the *Poisson bracket* on $\mathrm{Pol}^\bullet(T^*M) = \Gamma^\infty(M, S^\bullet TM)$ given in local coordinates by

$$\{f, g\} = \sum_{i=1}^n \Big(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}\Big).$$

(We recall that a Poisson bracket on a manifold Q is a Lie bracket $\{\cdot, \cdot\}$ on the algebra of functions $C^\infty(Q)$ such that for all $F \in C^\infty(M)$, the linear map $\{F, \cdot\}$ is a derivation of the algebra structure. One calls $\{F, \cdot\}$ the Hamiltonian vector field associated to F.) In this sense, the algebra $\mathfrak{D}^\bullet(M)$ of differential operators is regarded as a 'quantization' of the Poisson algebra $\mathrm{Pol}^\bullet(T^*M) = \Gamma^\infty(M, S^\bullet TM)$.

3.2. Graded Poisson algebras. To formalize this construction, we define a *graded Poisson algebra* of degree n to be a commutative graded algebra $\mathcal{P} = \bigoplus_{k\in\mathbb{Z}} \mathcal{P}^k$, together with a bilinear map $\{\cdot, \cdot\} \colon \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ (called Poisson bracket) such that

1. The space $\mathcal{P}[n]$ is a graded Lie algebra, with bracket $\{\cdot, \cdot\}$.
2. The map $f \mapsto \{f, \cdot\}$ defines a morphism of graded Lie algebras $\mathcal{P}[n] \to \mathrm{Der}_{\mathrm{alg}}(\mathcal{P})$.

That is, for any $f \in \mathcal{P}^k$, the map $\{f, \cdot\}$ is a degree $k - n$ derivation of the algebra structure. Note that the Poisson bracket is uniquely determined by its values on generators.

EXAMPLE 3.2. For any manifold M, the space $\Gamma^\infty(M, S^\bullet(TM))$ of fiberwise polynomial functions on T^*M is a graded Poisson algebra of degree 1.

EXAMPLE 3.3. Suppose $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ is any Lie algebra. Then the Lie bracket on $\mathfrak{g}$ extends to a Poisson bracket on the graded algebra $S(\mathfrak{g})$, making the latter into a graded Poisson algebra of degree 1. Viewing $S(\mathfrak{g})$ as polynomial functions on $\mathfrak{g}^*$, this is the *Kirillov Poisson structure* on $\mathfrak{g}^*$.

Conversely, if V is any vector space, then the structure of a graded Poisson algebra of degree 1 on $S(V)$ is *equivalent* to a Lie algebra structure on V.

Suppose now that $\mathcal{A}$ is a filtered algebra, with the property that the associated graded algebra $\mathrm{gr}(\mathcal{A})$ is commutative. This means that $[\mathcal{A}^{(k)}, \mathcal{A}^{(l)}] \subset \mathcal{A}^{(k+l-1)}$ for all k, l. Then the associated graded algebra $\mathcal{P} = \mathrm{gr}(\mathcal{A})$ becomes a graded Poisson algebra of degree 1, with bracket determined by a commutative diagram:

$$\begin{array}{ccc}
\mathcal{A}^{(k)} \otimes \mathcal{A}^{(l)} & \xrightarrow[{[\cdot,\cdot]}]{} & \mathcal{A}^{(k+l-1)} \\
\big\downarrow & & \big\downarrow \\
\mathcal{P}^{k} \otimes \mathcal{P}^{l} & \xrightarrow[\{\cdot,\cdot\}]{} & \mathcal{P}^{k+l-1}
\end{array}$$

We will think of $\mathcal{A}$ as a 'quantization' of $\mathcal{P} = \mathrm{gr}(\mathcal{A})$, on the feeble grounds that Poisson brackets correspond to commutators. For instance, as discussed later the enveloping algebra $U(\mathfrak{g})$ defines a quantization of the Poisson algebra $S(\mathfrak{g})$.

3.3. Graded super Poisson algebras. The symbol map for Clifford algebras may be put into a similar framework, but in a super-context. See [**48**] and [**20**]. See Appendix A for background on graded and filtered and super spaces.

DEFINITION 3.4. A *graded super Poisson algebra of degree n* is a commutative graded super algebra $\mathcal{P} = \bigoplus_{k\in\mathbb{Z}} \mathcal{P}^k$, together with a bilinear map $\{\cdot,\cdot\}\colon \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ such that

(1) The space $\mathcal{P}[n]$ is a graded super Lie algebra, with bracket $\{\cdot,\cdot\}$.
(2) The map $f \mapsto \{f, \cdot\}$ defines a morphism of graded super Lie algebras, $\mathcal{P}[n] \to \mathrm{Der}_{\mathrm{alg}}(\mathcal{P})$.

Here $\mathrm{Der}_{\mathrm{alg}}(\mathcal{P})$ signifies the derivations of $\mathcal{P}$ as a graded super algebra. Thus, the bracket $\{\cdot,\cdot\}$ is a map of degree $-n$, with the properties

$$\begin{aligned}
\{f_1, \{f_2, f_3\}\} &= \{\{f_1, f_2\}, f_3\} + (-1)^{(|f_1|-n)(|f_2|-n)}\{f_2, \{f_1, f_3\}\} \\
\{f_1, f_2\} &= -(-1)^{(|f_1|-n)\,(|f_2|-n)}\{f_2, f_1\}, \\
\{f_1, f_2 f_3\} &= \{f_1, f_2\} f_3 + (-1)^{(|f_1|-n)|f_2|} f_2\{f_1, f_3\}.
\end{aligned}$$

For $f \in \mathcal{P}^k$, the bracket $\{f, \cdot\}$ is a derivation of degree $k-n$ of the algebra and Lie algebra structures. Letting $\mathrm{Der}_{\mathrm{Poi}}(\mathcal{P})$ the derivations of $\mathcal{P}$ as a graded Poisson algebra, we hence have a morphism of graded super Lie algebras,

$$\mathcal{P}[n] \to \mathrm{Der}_{\mathrm{Poi}}(\mathcal{P}), \quad f \mapsto \{f, \cdot\}. \tag{29}$$

Note that $\mathcal{P}^n = \mathcal{P}[n]^0$ is an ordinary Lie algebra under the Poisson bracket.

Suppose now that $\mathcal{A}$ is a filtered super algebra such that the associated graded super algebra $\mathrm{gr}(\mathcal{A})$ is commutative. Thus $[\mathcal{A}^{(k)}, \mathcal{A}^{(l)}] \subset \mathcal{A}^{(k+l-1)}$.

Using the compatibility condition for the $\mathbb{Z}_2$-grading and filtration (cf. Appendix A)

$$(\mathcal{A}^{(2k)})^{\bar{0}} = (\mathcal{A}^{(2k+1)})^{\bar{0}}, \quad (\mathcal{A}^{(2k+1)})^{\bar{1}} = (\mathcal{A}^{(2k+2)})^{\bar{1}}, \tag{30}$$

we see that in fact

$$[\mathcal{A}^{(k)}, \mathcal{A}^{(l)}] \subset \mathcal{A}^{(k+l-2)}.$$

Hence the associated graded super algebra $\mathcal{P} = \mathrm{gr}(A)$ becomes a graded super Poisson algebra of degree 2, with Poisson bracket determined by the commutative diagram

$$\begin{tikzcd} \mathcal{A}^{(k)} \otimes \mathcal{A}^{(l)} \arrow[r, "{[\cdot,\cdot]}"'] \arrow[d] & \mathcal{A}^{(k+l-2)} \arrow[d] \\ \mathcal{P}^k \otimes \mathcal{P}^l \arrow[r, "{\{\cdot,\cdot\}}"'] & \mathcal{P}^{k+l-2} \end{tikzcd}$$

3.4. Poisson structures on $\wedge(V)$. Any symmetric bilinear form B on a vector space V induces on $\mathcal{A} = \wedge(V)$ the structure of a graded super Poisson algebra of degree 2. The Poisson bracket is given on generators $v, w \in V = \wedge^1(V)$ by

$$\{v, w\} = 2B(v, w).$$

In this way, one obtains a one-to-one correspondence between Poisson brackets (of degree -2) on $\wedge(V)$ and symmetric bilinear forms B. Clearly, this Poisson bracket is induced from the commutator on the Clifford algebra under the identification $\wedge(V) = \mathrm{gr}(\mathrm{Cl}(V; B))$ from Proposition 2.9.

Poisson bracket with elements of degree k defines derivations of degree $k-2$ of $\wedge(V)$. In particular, Poisson bracket with an element $v \in V$ is a derivations of degree -1, i.e. contractions:

$$\{v, \cdot\} = 2\iota(B^\flat(v)),$$

as one checks on generators $w \in V$. Similarly for $\lambda \in \wedge^2(V)$ we have

$$\{\lambda, \cdot\} = L_{A_\lambda},$$

since both sides are derivations given by $A_\lambda(w)$ on generators $w \in V$. In particular, the Lie bracket on $\wedge^2(V) = (\wedge(V)[2])^0$ defined by the Poisson bracket recovers our earlier definition as $\{\lambda, \lambda'\} = L_{A_\lambda}\lambda'$.

The graded Lie algebra $V \rtimes \wedge^2(V)$ from Section 2.11 is now interpreted as

$$(\wedge^1(V) \oplus \wedge^2(V))[2] \subset \wedge(V)[2].$$

As we had seen, the quantization map $q\colon \wedge(V) \to \mathrm{Cl}(V)$ restricts to a Lie algebra homomorphism on this Lie subalgebra. That is, on elements of degree ≤ 2 the quantization map takes Poisson brackets to commutators. This is no longer true, in general, for higher order elements.

EXAMPLE 3.5. Let $\phi \in \wedge^3(V)$, so that $\{\phi,\phi\} \in \wedge^4(V)$. As we saw in Example 2.11,

$$[q(\phi), q(\phi)] = 2q(\phi)^2 \in q(\wedge^0(V) \oplus \wedge^4(V)).$$

The leading term is given by the Poisson bracket, that is,

$$[q(\phi), q(\phi)] - q(\{\phi,\phi\}) \in \wedge^0(V) = \mathbb{K}.$$

In general, this scalar is non-zero. For instance, if $V = \mathbb{R}^3$ with the standard bilinear form, and $\phi = e_1 \wedge e_2 \wedge e_3$ (the volume element) then

$$[q(\phi), q(\phi)] = 2q(\phi)^2 = 2(e_1e_2e_3)^2 = -2.$$

More generally, suppose V is a finite-dimensional vector space, with non-degenerate symmetric bilinear form B, and let e_a be a basis of V, with B-dual basis e^a. Given $\phi \in \wedge^3(V)$, define its components in the two bases by

$$\phi = \frac{1}{6}\sum_{abc} \phi^{abc} e_a \wedge e_b \wedge e_c = \frac{1}{6}\sum_{abc} \phi_{abc} e^a \wedge e^b \wedge e^c.$$

According to Proposition 2.21, the constant term in $[q(\phi), q(\phi)] = 2q(\phi)^2$ is obtained by applying the operator $\frac{1}{3!}(-\sum_a \iota(e^a) \otimes \iota(e_a))^3$ to $2\phi \otimes \phi \in \wedge(V) \otimes \wedge(V)$. This gives

$$\begin{aligned} [q(\phi), q(\phi)] - q(\{\phi,\phi\}) &= \frac{1}{3}\Big(-\sum_a \iota(e^a) \otimes \iota(e_a)\Big)^3 (\phi \otimes \phi) \\ &= -\frac{1}{3}\sum_{abc} \phi^{abc}\phi_{abc}. \end{aligned}$$

It can be shown (cf. §6, Proposition 7.1 below) that $\{\phi,\phi\} = 0$ if and only if the formula $[v,w] := \{\{\phi, v\}, w\}$ defines a Lie bracket on V.

Consider again the map $\wedge(V)[n] \to \mathrm{Der}_{\mathrm{Poi}}(\wedge(V)),\ f \mapsto \{f, \cdot\}$ from (29). Its composition with the inclusion $\mathrm{Der}_{\mathrm{Poi}}(\wedge(V)) \to \mathrm{Der}_{\mathrm{alg}}(\wedge(V)) \cong \wedge(V) \otimes V^*$ is given by

$$f \mapsto (-1)^{|f|} 2 \sum_a \iota(e^a) f \otimes B^\flat(e_a),$$

where e_a is a basis of V with dual basis $e^a \in V^*$. To verify this identity, it suffices to evaluate on generators $v \in V$: Indeed,

$$\{f, v\} = (-1)^{|f|}\{v, f\} = (-1)^{|f|} 2\iota(B^\flat(v)) f = (-1)^{|f|} 2 \sum_a B(e_a, v)\, \iota(e^a) f.$$

If B is non-degenerate, then all derivations of the Poisson structure on $\wedge(V)$ are inner.

PROPOSITION 3.6. *Suppose B is a non-degenerate symmetric bilinear form on V. Then the map*

$$\wedge(V)[n] \to \mathrm{Der}_{\mathrm{Poi}}(\wedge(V))$$

is surjective, with kernel the scalars $\mathbb{K}[n]$.

Proof. Suppose $D \in \mathrm{Der}_{\mathrm{Poi}}(\wedge(V))^m$ is a derivation of degree $m \geq -1$ of the Poisson structure. Let $e_1, \ldots, e_n$ be a basis of V, with e^i the B-dual basis. Then $De_i \in \wedge^{m+1}(V)$, and

$$\{De_i,\ e_j\} + \{De_j, e_i\} = \{De_i,\ e_j\} + (-1)^m\{e_i, De_j\} = D\{e_i, e_j\} = 0.$$

As a consequence,

$$\begin{aligned}\{\sum_j De_j \wedge e^j, e_i\} &= \sum_j De_j \wedge \{e^j, e_i\} - \sum_j \{De_j,\ e_i\} \wedge e^j \\ &= 2De_i + \sum_j \{De_i, e_j\} \wedge e^j \\ &= 2(m+2)De_i\end{aligned}$$

where we used $\sum_j \{e_I, e_j\} \wedge e^j = 2|I|e_I$. It follows that D is Poisson bracket with the element

$$\frac{1}{2(m+2)} \sum_j De_j \wedge e^j.$$

$\square$

CHAPTER 3

The spin representation

The Clifford algebra for a vector space V with split bilinear form B has an (essentially) unique irreducible module, called the *spinor module* S. The Clifford action restricts to a representation of the *Spin group* $\mathrm{Spin}(V)$, known as the *spin representation*. After developing the basic properties of spinor modules and the spin representation, we give a discussion of *pure spinors* and their relation with Lagrangian subspaces. Throughout, we will assume that V is a finite-dimensional vector space over a field $\mathbb{K}$ of characteristic zero, and that the bilinear form B on V is non-degenerate. We will write $\mathrm{Cl}(V)$ in place of $\mathrm{Cl}(V;B)$.

1. The Clifford group and the spin group

1.1. The Clifford group. Recall that $\Pi\colon \mathrm{Cl}(V)\to\mathrm{Cl}(V),\ x\mapsto(-1)^{|x|}x$ denotes the parity automorphism of the Clifford algebra. Let $\mathrm{Cl}(V)^\times$ be the group of invertible elements in $\mathrm{Cl}(V)$.

Definition 1.1. The *Clifford group* $\Gamma(V)$ is the subgroup of $\mathrm{Cl}(V)^\times$, consisting of all $x\in\mathrm{Cl}(V)^\times$ such that

$$A_x(v):=\Pi(x)vx^{-1}\in V$$

for all $v\in V\subset\mathrm{Cl}(V)$.

Hence, by definition the Clifford group comes with a natural representation, $\Gamma(V)\to\mathrm{GL}(V),\ x\mapsto A_x$. Let $S\Gamma(V)=\Gamma(V)\cap\mathrm{Cl}^{\bar 0}(V)^\times$ denote the *special Clifford group*.

Theorem 1.2. *The natural representation of the Clifford group takes values in* $\mathrm{O}(V)$*, and defines an exact sequence,*

$$1\longrightarrow\mathbb{K}^\times\longrightarrow\Gamma(V)\longrightarrow\mathrm{O}(V)\longrightarrow 1.$$

It restricts to a similar exact sequence for the special Clifford group,

$$1\longrightarrow\mathbb{K}^\times\longrightarrow S\Gamma(V)\longrightarrow\mathrm{SO}(V)\longrightarrow 1.$$

The elements of $\Gamma(V)$ *are all products*

$$x=v_1\cdots v_k \tag{31}$$

where $v_1,\ldots,v_k\in V$ *are non-isotropic. The corresponding element* A_x *is a product of reflections:*

$$A_x=R_{v_1}\cdots R_{v_k}. \tag{32}$$

In particular, every element $x \in \Gamma(V)$ has a definite parity (given by the parity of k in (31)*), and $S\Gamma(V)$ consists of products* (31) *with k even.*

Proof. Let $x \in \mathrm{Cl}(V)$. The transformation A_x is trivial if and only if $\Pi(x)v = vx$ for all $v \in V$, i.e. if and only if $[v, x] = 0$ for all $v \in V$. That is, it is the intersection of the center $\mathbb{K} \subset \mathrm{Cl}(V)$ with $\Gamma(V)$. (See §2 Lemma 2.17.) This shows that the kernel of the homomorphism $\Gamma(V) \to \mathrm{GL}(V),\ x \mapsto A_x$ is the group $\mathbb{K}^\times$ of invertible scalars.

Applying $-\Pi$ to the definition of A_x, we obtain $A_x(v) = xv\Pi(x)^{-1} = A_{\Pi(x)}(v)$. This shows $A_{\Pi(x)} = A_x$ for $x \in \Gamma(V)$. Thus $\Pi(x)$ is a scalar multiple of x, in fact $\Pi(x) = \pm x$ since Π is the parity operator. This shows that elements of $\Gamma(V)$ have definite parity. For $x \in \Gamma(V)$ and $v, w \in V$ we have, using again $A_{\Pi(x)} = A_x$,

$$\begin{aligned}
2B(A_x(v), A_x(w)) &= A_x(v)A_x(w) + A_x(w)A_x(v) \\
&= A_x(v)A_{\Pi(x)}(w) + A_x(w)A_{\Pi(x)}(v) \\
&= \Pi(x)(vw + wv)\Pi(x^{-1}) \\
&= 2B(v, w)\Pi(x)\Pi(x^{-1}) \\
&= 2B(v, w).
\end{aligned}$$

This proves that $A_x \in \mathrm{O}(V)$ for all $x \in \Gamma(V)$. Suppose now that $v \in V$ is non-isotropic. Then it is invertible in the Clifford algebra, with $v^{-1} = v/B(v, v)$ and $\Pi(v) = -v$. For all $w \in V$,

$$A_v(w) = -vwv^{-1} = (wv - 2B(v, w))v^{-1} = w - 2\frac{B(v, w)}{B(v, v)}v = R_v(w).$$

Hence $v \in \Gamma(V)$, with $A_v = R_v$ the reflection defined by v. More generally, this proves (32) whenever x is of the form (31). By the E. Cartan-Dieudonné Theorem 4.5, any $A \in \mathrm{O}(V)$ is a product of reflections R_{v_i}. This shows the map $x \mapsto A_x$ is onto $\mathrm{O}(V)$, and that $\Gamma(V)$ is generated by the non-isotropic vectors in V. The remaining statements are clear. □

Since every $x \in \Gamma(V)$ can be written in the form (31), it follows that the element $x^\top x$ lies in $\mathbb{K}^\times$. This defines the *norm homomorphism*

$$\mathsf{N}\colon \Gamma(V) \to \mathbb{K}^\times,\ x \mapsto x^\top x. \tag{33}$$

It is a group homomorphism, and has the property

$$\mathsf{N}(\lambda x) = \lambda^2 \mathsf{N}(x)$$

for $\lambda \in \mathbb{K}^\times$.

Example 1.3. The chirality element $\Gamma \in \mathrm{Cl}(V)$ defined by a choice of generator $\Gamma_\wedge \in \det(V)$ is an element of the Clifford group $\Gamma(V)$, and is contained on $S\Gamma(V)$ if and only if $\dim V = 2m$ is even. In the special case of a vector space with split bilinear form, and Γ normalized so that $\Gamma^2 = 1$ (see Equation (20)), one has

$$\mathsf{N}(\Gamma) = \Gamma^\top \Gamma = (-1)^m.$$

Example 1.4. Consider $V = F^* \oplus F$ with $\dim F = 1$. Choose dual generators $e \in F$, $f \in F^*$ so that $B(e, f) = \frac{1}{2}$. One checks that an even element $x = s + tfe \in \mathrm{Cl}^{\bar{0}}(V)$ with $s, t \in \mathbb{K}$ lies in $S\Gamma(V)$ if and only if $s, s + t$ are both invertible, and in that case

$$A_x(e) = \tfrac{s}{s+t}\, e, \quad A_x(f) = \tfrac{s+t}{s}\, e.$$

We have $\mathsf{N}(x) = x^\top x = s(s + t)$.

1.2. The groups $\mathrm{Pin}(V)$ **and** $\mathrm{Spin}(V)$**.** We make the following definitions.

Definition 1.5. The *Pin group* $\mathrm{Pin}(V)$ is the kernel of the norm homomorphism $\mathsf{N}\colon \Gamma(V) \to \mathbb{K}^\times$. Its intersection with $S\Gamma(V)$ is called the *Spin group*, and is denoted $\mathrm{Spin}(V)$.

The normalization $\mathsf{N}(x) = 1$ specifies $x \in \Gamma(V)$ up to sign. Hence one obtains exact sequences,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(V) \longrightarrow \mathrm{O}(V),$$
$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V).$$

In general, the maps to $\mathrm{SO}(V)$, $\mathrm{O}(V)$ need not be surjective. A sufficient condition for surjectivity is that every element in $\mathbb{K}$ admits a square root, since one may then rescale any $x \in \Gamma(V)$ so that $\mathsf{N}(x) = 1$. Theorem 1.2 shows that in this case, $\mathrm{Pin}(V)$ is the set of products $v_1 \cdots v_k$ of elements $v_i \in V$ with $B(v_i, v_i) = 1$, while $\mathrm{Spin}(V)$ consists of similar products with k even.

Remark 1.6. If V has non-zero isotropic vectors, then the condition that all elements in $\mathbb{K}$ have square roots is also necessary. Indeed, let $e \neq 0$ be isotropic, and let f be an isotropic vector with $B(e, f) = \frac{1}{2}$. Let $A \in \mathrm{SO}(V)$ be equal to the identity on $\mathrm{span}\{e, f\}^\perp$, and

$$A(e) = r\, e, \quad A(f) = r^{-1}\, f$$

with $r \in \mathbb{K}^\times$. As shown in Example 1.4, the lifts of A to $S\Gamma(V)$ are elements of the form $x = s + tfe$ with $r = s(s + t)^{-1}$. Since $\mathsf{N}(x) = s(s + t) = \frac{s^2}{r}$ we see that $x \in \mathrm{Spin}(V)$ if and only if $r = s^2$. The choice of square root of r specifies the lift x.

Remark 1.7. If $\mathbb{K} = \mathbb{R}$, the Pin and Spin groups are sometimes defined using a weaker condition $\mathsf{N}(x) = \pm 1$. This then guarantees that the maps to $\mathrm{O}(V), \mathrm{SO}(V)$ are surjective.

For $\mathbb{K} = \mathbb{R}$, $V = \mathbb{R}^{n,m}$ we use the notation

$$\mathrm{Pin}(n, m) = \mathrm{Pin}(\mathbb{R}^{n,m}), \quad \mathrm{Spin}(n, m) = \mathrm{Spin}(\mathbb{R}^{n,m}).$$

If $m = 0$ we simply write $\mathrm{Pin}(n) = \mathrm{Pin}(n, 0)$ and $\mathrm{Spin}(n) = \mathrm{Spin}(n, 0)$.

THEOREM 1.8. *Let* $\mathbb{K} = \mathbb{R}$. *Then* $\mathrm{Spin}(n,m)$ *is a double cover of the identity component* $\mathrm{SO}_0(n,m)$. *If* $n \geq 2$ *or* $m \geq 2$, *the group* $\mathrm{Spin}(n,m)$ *is connected.*

PROOF. The cases of $(n,m) = (0,1),\ (1,0)$ are trivial. If $(n,m) = (1,1)$ one has $\mathrm{SO}_0(1,1) = \mathbb{R}_{>0}$, and $\mathrm{Spin}(1,1) = \mathbb{Z}_2 \times \mathbb{R}_{>0}$ (see Example 1.4). Suppose $n \geq 2$ or $m \geq 2$. To show that $\mathrm{Spin}(n,m)$ is connected, it suffices to show that the elements ± 1 (the pre-image of the group unit in $\mathrm{SO}(n,m)$) are in the same connected component. Let

$$v(\theta) \in \mathbb{R}^{n,m}, \quad 0 \leq \theta \leq \pi$$

be a continuous family of non-isotropic vectors with the property $v(\pi) = -v(0)$. Such a family exists, since V contains a 2-dimensional subspace isomorphic to $\mathbb{R}^{2,0}$ or $\mathbb{R}^{0,2}$. Rescale the vectors $v(\theta)$ to satisfy $B(v(\theta), v(\theta)) = \pm 1$. Then $[0,\pi] \to \mathrm{Spin}(n,m),\ \theta \mapsto x(\theta) = v(\theta)v(0)$ is a path connecting $+1$ and -1. $\square$

The groups $\mathrm{Spin}(n,m)$ are usually not simply connected. Indeed since since $\mathrm{SO}_0(n,m)$ has maximal compact subgroup $\mathrm{SO}(n) \times \mathrm{SO}(m)$, the fundamental group is

$$\pi_1(\mathrm{SO}_0(n,m)) = \pi_1(\mathrm{SO}(n)) \times \pi_1(\mathrm{SO}(m))$$

In particular, if $n, m > 2$ the fundamental group of $\mathrm{SO}(n,m)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, and hence that of its double cover $\mathrm{Spin}(n,m)$ is $\mathbb{Z}_2$. The spin group is simply connected only in the cases (n,m) with $n > 2$ and $m = 0,1$, or $n = 0,1$ and $m > 2$, and only in those cases $\mathrm{Spin}(n,m)$ the universal cover of $\mathrm{SO}_0(n,m)$. Of particular interest is the case $m = 0$, where $\mathrm{Spin}(n)$ defines the universal cover of $\mathrm{SO}(n)$ for $n > 2$. In low dimensions, one has the exceptional isomorphisms

$$\begin{aligned}
\mathrm{Spin}(2) &= \mathrm{SO}(2),\\
\mathrm{Spin}(3) &= \mathrm{SU}(2),\\
\mathrm{Spin}(4) &= \mathrm{SU}(2) \times \mathrm{SU}(2),\\
\mathrm{Spin}(5) &= \mathrm{Sp}(2),\\
\mathrm{Spin}(6) &= \mathrm{SU}(4).
\end{aligned}$$

Here $\mathrm{Sp}(n)$ is the compact symplectic group, i.e. the group of norm-preserving automorphisms of the n-dimensional quaternionic vector space $\mathbb{H}^n$. The isomorphisms for $\mathrm{Spin}(3), \mathrm{Spin}(4)$ follow from §1 Proposition 6.3, while the isomorphisms for $\mathrm{Spin}(5)$, $\mathrm{Spin}(6)$ are obtained from a discussion of the spin representation of these groups, see Section 7.6 below. For $n \geq 7$, there are no further accidental isomorphisms of this type.

Let us now turn to the case $\mathbb{K} = \mathbb{C}$, so that $V \cong \mathbb{C}^n$ with the standard bilinear form. We write $\mathrm{Pin}(n,\mathbb{C}) = \mathrm{Pin}(\mathbb{C}^n)$ and $\mathrm{Spin}(n,\mathbb{C}) = \mathrm{Spin}(\mathbb{C}^n)$.

PROPOSITION 1.9. $\mathrm{Pin}(n,\mathbb{C})$ *and* $\mathrm{Spin}(n,\mathbb{C})$ *are double covers of* $\mathrm{O}(n,\mathbb{C})$ *and* $\mathrm{SO}(n,\mathbb{C})$. *Furthermore,* $\mathrm{Spin}(n,\mathbb{C})$ *is connected and simply connected,* *i.e. it is the universal cover of* $\mathrm{SO}(n, \mathbb{C})$*. The group* $\mathrm{Spin}(n)$ *is the maximal compact subgroup of* $\mathrm{Spin}(n, \mathbb{C})$*.*

PROOF. The first part is clear, since for $x \in \Gamma(\mathbb{C}^n)$ the condition $N(\lambda x) = 1$ determines λ up to a sign. The second part follows by the same argument as in the real case, or alternatively by observing that ± 1 are in the same component of $\mathrm{Spin}(n, \mathbb{R}) \subset \mathrm{Spin}(n, \mathbb{C})$. Finally, since $\mathrm{SO}(n)$ is the maximal compact subgroup of $\mathrm{SO}(n, \mathbb{C})$, its pre-image $\mathrm{Spin}(n)$ is the maximal compact subgroup of $\mathrm{Spin}(n, \mathbb{C})$. $\square$

Suppose V is a vector over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with non-degenerate symmetric bilinear form B. Since $\mathrm{Spin}(V)$ is a double cover of the identity component of $\mathrm{SO}(V)$, its Lie algebra is $\mathfrak{o}(V)$. The following result realizes the exponential map for $\mathrm{Spin}(V)$ directly in terms of the Clifford algebra. View $\mathrm{Cl}^{\bar{0}}(V)$ as a Lie algebra under Clifford commutator, with corresponding Lie group $\mathrm{Cl}^{\bar{0}}(V)^\times$. The exponential map $\exp\colon \mathrm{Cl}^{\bar{0}}(V) \to \mathrm{Cl}^{\bar{0}}(V)^\times$ for this Lie group is given as a power series. Recall the Lie algebra homomorphism $\gamma\colon \mathfrak{o}(V) \to \mathrm{Cl}^{\bar{0}}(V)$ from §2.11, see (25).

PROPOSITION 1.10. *The following diagram commutes:*

$$\begin{tikzcd} \mathrm{Spin}(V) \arrow[r] & \mathrm{Cl}^{\bar{0}}(V)^\times \\ \mathfrak{o}(V) \arrow[u, "\exp"] \arrow[r, "\gamma"'] & \mathrm{Cl}^{\bar{0}}(V) \arrow[u, "\exp"] \end{tikzcd}$$

PROOF. For $A \in \mathfrak{o}(V)$ we have $A(v) = [\gamma(A), v]$ for $v \in V$, and accordingly

$$\exp(A)(v) = \exp(\mathrm{ad}(\gamma(A))v$$

Using the identity $\exp(a)b\exp(-a) = \exp(\mathrm{ad}(a))b$ for elements a, b in a finite-dimensional (ordinary) algebra, we obtain

$$\exp(A)(v) = e^{\gamma(A)} v e^{-\gamma(A)}.$$

Since the left hand side lies in V, this shows $e^{\gamma(A)} \in S\Gamma(V)$ by definition of the Clifford group. Furthermore, since $\gamma(A)^\top = -\gamma(A)$ we have

$$(e^{\gamma(A)})^\top = e^{\gamma(A)^\top} = e^{-\gamma(A)},$$

and therefore $N(e^{\gamma(A)}) = 1$. That is,

$$e^{\gamma(A)} \in \mathrm{Spin}(V)$$

This shows that the group $\mathrm{Spin}(V) \subset \mathrm{Cl}(V)^\times$ has Lie algebra $\gamma(\mathfrak{o}(V)) \subset \mathrm{Cl}^{\bar{0}}(V)$. $\square$

EXAMPLE 1.11. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $V = \mathbb{K}^2$ with the standard bilinear form. Consider the element $A \in \mathfrak{o}(V)$ defined by $\lambda(A) = e_1 \wedge e_2$. Then $\gamma(A) = e_1 e_2$. Since $(e_1 e_2)^2 = -1$, the 1-parameter group of elements

$$x(\theta) = \exp(\theta/2 e_1 e_2) \in \mathrm{Spin}(V)$$

is given by the formula,

$$x(\theta) = \cos(\theta/2) + \sin(\theta/2) e_1 e_2.$$

To find its action $A_{x(\theta)}$ on V, we compute

$$\begin{aligned} x(\theta) e_1 x(-\theta) &= (\cos(\theta/2) + \sin(\theta/2) e_1 e_2) e_1 (\cos(\theta/2) - \sin(\theta/2) e_1 e_2) \\ &= (\cos(\theta/2) e_1 - \sin(\theta/2) e_2)(\cos(\theta/2) - \sin(\theta/2) e_1 e_2) \\ &= (\cos^2(\theta/2) - \sin^2(\theta/2)) e_1 - 2 \sin(\theta/2)\cos(\theta/2) e_2 \\ &= \cos(\theta) e_1 - \sin(\theta) e_2 \end{aligned}$$

This verifies that $A_{x(\theta)}$ is given as rotations by θ. We see explicitly that $A_{x(\theta+2\pi)} = A_{x(\theta)}$ while $x(\theta + 2\pi) = -x(\theta)$.

2. Clifford modules

2.1. Basic constructions. Let V be a vector space with symmetric bilinear form B, and $\mathrm{Cl}(V)$ the corresponding Clifford algebra. A module over the super algebra $\mathrm{Cl}(V)$ is called a *Clifford module*, or simply $\mathrm{Cl}(V)$-module. That is, a Clifford module is a finite-dimensional super vector space E together with a morphism of super algebras,

$$\varrho_E \colon \mathrm{Cl}(V) \to \mathrm{End}(E).$$

Equivalently, a Clifford module is given by a linear map $\varrho_E \colon V \to \mathrm{End}^{\bar{1}}(E)$ such that

$$\varrho_E(v)\varrho_E(w) + \varrho_E(w)\varrho_E(v) = 2\, B(v, w) 1$$

for all $v, w \in V$. A morphism of Clifford modules E, E' is a morphism of super vector spaces $f \colon E \to E'$ intertwining the Clifford actions.

Remark 2.1. If E carries a filtration, compatible with the $\mathbb{Z}_2$-grading in the sense of Appendix A and such that the Clifford action is filtration preserving, then we call E a *filtered* Clifford module. To construct a compatible filtration on a given Clifford module, choose any subspace $E' \subset E$ of definite parity, and pick $l \in \mathbb{Z}$, even or odd depending on the parity of E'. Then put $E^{(l+m)} = \mathrm{Cl}(V)^{(m)} E'$ for $m \in \mathbb{Z}$.

Remark 2.2. One can also consider modules over $\mathrm{Cl}(V)$, viewed as an *ordinary* (rather than super) algebra. These will be referred to as *ungraded Clifford modules.*

There are several standard constructions with Clifford modules:

1. **Submodules, quotient modules.** A submodule of a $\mathrm{Cl}(V)$-module E is a super subspace E_1 which is stable under the module action. In this case, the quotient E/E_1 becomes a $\mathrm{Cl}(V)$- module in an obvious way. A $\mathrm{Cl}(V)$-module E is called *irreducible* if there are no submodules other than E and $\{0\}$.
2. **Direct sum.** The direct sum of two $\mathrm{Cl}(V)$-modules E_1, E_2 is again a $\mathrm{Cl}(V)$-module, with $\varrho_{E_1 \oplus E_2} = \varrho_{E_1} \oplus \varrho_{E_2}$.

(3) **Dual modules.** If E is any Clifford module, the dual space $E^* = \mathrm{Hom}(E, \mathbb{K})$ becomes a Clifford module, with module structure defined in terms of the canonical anti-automorphism of $\mathrm{Cl}(V)$ by

$$\varrho_{E^*}(x) = \varrho_E(x^\top)^*, \ x \in \mathrm{Cl}(V).$$

That is, $\langle \varrho_{E^*}(x)\psi, \beta\rangle = \langle \psi, \varrho_E(x^\top)\beta\rangle$ for $\psi \in E^*$ and $\beta \in E$. If E is a filtered $\mathrm{Cl}(V)$-module, then E^* with the dual filtration (see Appendix A) is again a filtered $\mathrm{Cl}(V)$-module.

(4) **Tensor products.** Suppose V_1, V_2 are vector spaces with symmetric bilinear forms B_1, B_2. If E_1 is a $\mathrm{Cl}(V_1)$-module and E_2 is a $\mathrm{Cl}(V_2)$-module, the tensor product $E_1 \otimes E_2$ is a module over $\mathrm{Cl}(V_1) \otimes \mathrm{Cl}(V_2) = \mathrm{Cl}(V_1 \oplus V_2)$, with

$$\varrho_{E_1 \otimes E_2}(x_1 \otimes x_2) = \varrho_{E_1}(x_1) \otimes \varrho_{E_2}(x_2).$$

In particular, $\mathrm{Cl}(V)$-modules E can be tensored with super vector spaces, viewed as modules over the Clifford algebra for the trivial vector space $\{0\}$.

(5) **Opposite grading.** If E is any $\mathrm{Cl}(V)$-module, then the same space E with the opposite $\mathbb{Z}_2$-grading is again a $\mathrm{Cl}(V)$-module, denoted E^{op}.

Given a $\mathrm{Cl}(V)$-module E, one obtains a group representation of the Clifford group $\Gamma(V)$ by restriction, and $S\Gamma(V)$ acquires two representations $E^{\bar{0}}$, $E^{\bar{1}}$.

The first example of a Clifford module is the Clifford algebra $\mathrm{Cl}(V)$ itself, with module structure given by multiplication from the left. The exterior algebra $\wedge(V)$ is a Clifford module, with action given on generators by (17). The symbol map $\sigma\colon \mathrm{Cl}(V) \cong \wedge(V)$ from §2 Section 2.5 is characterized as the unique isomorphism of Clifford modules taking $1 \in \mathrm{Cl}(V)$ to $1 \in \wedge(V)$. The Clifford module $\mathrm{Cl}(V) \cong \wedge(V)$ is self-dual:

PROPOSITION 2.3. *The* $\mathrm{Cl}(V)$*-module* $E = \mathrm{Cl}(V)$ *(with action by left multiplication) is canonically isomorphic to its dual.*

PROOF. The map

$$\mathrm{Cl}(V) \to \mathrm{Cl}(V)^*, \ y \mapsto \phi_y,$$

where $\langle \phi_y, z\rangle = \mathrm{tr}(y^\top z)$, is a linear isomorphism of super spaces. For $x \in \mathrm{Cl}(V)$ we have,

$$\langle \phi_{xy},\, z\rangle = \mathrm{tr}(y^\top x^\top z) = \langle \phi_y,\, x^\top z\rangle = \langle x.\phi_y, z\rangle$$

hence ϕ is $\mathrm{Cl}(V)$-equivariant. □

Note however that the isomorphism described above does *not* preserve filtrations.

2.2. The spinor module S_F. Let V be a vector space of dimension $n = 2m$, equipped with a split bilinear form B. View $\mathrm{Cl}(V)$ as a Clifford module under multiplication from the left, and let $F \subset V$ be a Lagrangian subspace. Then the left-ideal $\mathrm{Cl}(V)F$ is a submodule of $\mathrm{Cl}(V)$. The *spinor module associated to F* is the quotient $\mathrm{Cl}(V)$-module,

$$\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F.$$

The spinor module S_F may also be viewed as an *induced module.* View $\wedge(F) = \mathrm{Cl}(F)$ as a subalgebra of $\mathrm{Cl}(V)$, by the natural homomorphism extending the inclusion $F \subset V$.

PROPOSITION 2.4. *Let $\mathbb{K}$ be the trivial $\wedge(F)$-module, that is*

$$\phi.t = \phi_{[0]}t, \qquad \phi \in \wedge(F),\ t \in \mathbb{K}.$$

Then S_F is the corresponding induced module:

$$\mathsf{S}_F = \mathrm{Cl}(V) \otimes_{\wedge(F)} \mathbb{K}.$$

PROOF. By definition of the tensor product over $\wedge(F)$, the right hand side is the quotient of $\mathrm{Cl}(V) \otimes \mathbb{K}$ by the subspace generated by all $x \otimes \phi.t - x\phi \otimes t$. But this is the same as the subspace of $\mathrm{Cl}(V)$ by the subspace generated by all $x(\phi - \phi_{[0]})$ for $\phi \in \wedge(F)$, which is exactly $\mathrm{Cl}(V)F$. $\square$

PROPOSITION 2.5. *The choice of a Lagrangian complement $F' \cong F^*$ to F identifies*

$$\mathsf{S}_F = \wedge(F^*)$$

with Clifford action given on generators by $\varrho(\mu, v) = \epsilon(\mu) + \iota(v)$ for $v \in F,\ \mu \in F^*$.

PROOF. The choice of F' identifies $V = F^* \oplus F$, with the bilinear form $B((\mu_1, v_1), (\mu_2, v_2)) = \frac{1}{2}(\langle \mu_1, v_2\rangle + \langle \mu_2, v_1\rangle)$. Both $\wedge(F)$ and $\wedge(F^*)$ are embedded as subalgebras of $\mathrm{Cl}(V)$, and the multiplication map defines a homomorphism of filtered super vector spaces

$$\wedge(F^*) \otimes \wedge(F) \to \mathrm{Cl}(F^* \oplus F). \tag{34}$$

The associated graded map is the isomorphism $\wedge(F^*) \otimes \wedge(F) \to \wedge(F^* \oplus F)$ given by wedge product. Hence (34) is a linear isomorphism. Under this identification, $\mathrm{Cl}(V)F = \wedge(F^*) \otimes \bigoplus_{k\geq 1} \wedge^k(F)$, which has a natural complement $\wedge(F^*)$. Consequently

$$\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F \cong \wedge(F^*).$$

The Clifford action of $(\mu, v) \in F^* \oplus F$ on any $\psi \in \wedge(F^*)$ is given by Clifford multiplication by $\mu + v$ from the left, followed by projection to $\wedge(F^*)$ along $\mathrm{Cl}(V)F$. Since

$$(\mu + v)\psi = \mu\psi + [v, \psi] + (-1)^{|\psi|}\psi v = (\mu \wedge \psi + \iota(v)\psi) + (-1)^{|\psi|}\psi v,$$

and $\psi v \in \mathrm{Cl}(V)F$, this confirms our description of the action on $\wedge(F^*)$. $\square$

The spinor module S_F carries a canonical filtration, compatible with the $\mathbb{Z}_2$-grading, given as the quotient of the filtration of the Clifford algebra:

$$\mathsf{S}_F^{(k)} = \mathrm{Cl}(V)^{(k)}/\mathrm{Cl}(V)^{(k-1)}F.$$

PROPOSITION 2.6. (1) *The associated graded space for the filtration on the spinor module is*

$$\mathrm{gr}(\mathsf{S}_F) = \wedge(F^*).$$

(2) *For $v \in V$, the operator $\varrho(v)$ on S_F has filtration degree -1, and the associated graded operator $\mathrm{gr}^1(\varrho(v))$ on $\mathrm{gr}(\mathsf{S}_F) = \wedge(F^*)$ is wedge product with the image of v in $F^* = V/F$.*

(3) *For $v \in F \subset V$, the operator $\varrho(v)$ has filtration degree -1, and the associated graded operator is contraction: $\mathrm{gr}^{-1}(\varrho(v)) = \iota(v)$ That is, if $\phi \in \mathsf{S}_F^{(k)}$ the leading term of $\varrho(v)\phi \in \mathsf{S}_F^{(k-1)}$, is*

$$\mathrm{gr}^{k-1}(\varrho(v)\phi) = \iota(v)\,\mathrm{gr}^k(\phi).$$

PROOF. Since $\mathrm{gr}(\mathrm{Cl}(V)) = \wedge(V)$, the associated graded space is

$$\mathrm{gr}(\mathsf{S}_F) = \wedge(V)/\wedge(V)F = \wedge(V/F) \cong \wedge(F^*).$$

Choose a Lagrangian complement F' to identify $\mathsf{S}_F = \wedge(F^*)$. Then $\mathsf{S}_F^{(k)} = \bigoplus_{i\le k}\wedge^i F^*$, and the remaining claims are immediate from Proposition 2.5. □

2.3. The dual spinor module S^F. We define a *dual spinor module associated to F*

$$\mathsf{S}^F = \mathrm{Cl}(V)\det(F)$$

where $\det(F) = \wedge^m(F)$ is the determinant line.

PROPOSITION 2.7. *The choice of a Lagrangian complement $F' \cong F^*$ to F identifies*

$$\mathsf{S}^F = \wedge(F)$$

with Clifford action given on generators by $\varrho(\mu, v) = \iota(\mu) + \epsilon(v)$.

PROOF. Following the notation from the proof of Proposition 2.5, we have

$$\mathsf{S}^F = \mathrm{Cl}(V)\det(F) = \wedge(F^*)\otimes\det(F) \cong \wedge(F),$$

where the isomorphism is given by the contraction homomorphism $\wedge(F^*) \to \mathrm{End}(\wedge(F))$. One readily checks that this identification takes the Clifford action to $\iota(\mu) + \epsilon(v)$. □

Propositions 2.5 and 2.7 suggest that the spinor modules $\mathsf{S}_F, \mathsf{S}^F$ are in duality. In fact, this duality does does not depend on the choice of complement.

PROPOSITION 2.8. *There is a non-degenerate pairing*

$$\mathsf{S}^F \times \mathsf{S}_F \to \mathbb{K}, \quad (y,[x]) \mapsto \mathrm{tr}(y^\top x)$$

for $y \in \mathsf{S}^F \subset \mathrm{Cl}(V)$ *and* $[x] \in \mathrm{Cl}(V)/\mathrm{Cl}(V)F$ *(represented by an element* $x \in \mathrm{Cl}(V)$*). The pairing satisfies*

$$(y,\ \varrho(z)[x]) = (\varrho(z^T)y, [x]),$$

hence it defines an isomorphism of Clifford modules $\mathsf{S}^F \cong \mathsf{S}_F^*$. *Choosing a Lagrangian complement* F' *to identify* $\mathsf{S}_F = \wedge(F^*)$ *and* $\mathsf{S}^F = \wedge(F)$*, the pairing is just the usual pairing between* $\wedge(F^*)$ *and* $\wedge(F)$*.*

PROOF. The pairing is well-defined, since $x_1 y^\top = 0$ for $y \in \mathrm{Cl}(V)\det(F)$ and $x_1 \in \mathrm{Cl}(V)F$, hence $\mathrm{tr}(y^\top x_1) = 0$. The pairing satisfies

$$(y,\ \varrho(z)[x]) = \mathrm{tr}(y^\top z x) = \mathrm{tr}((z^\top y)^\top x) = (\varrho(z^T)y, [x]).$$

Choose a Lagrangian complement F' to F, and view S_F as a subspace $\wedge(F^*) \subset \mathrm{Cl}(V)$ as in (34). The pairing between $[x] = \phi \in \mathsf{S}_F$ and $y = \phi'\chi \in \mathsf{S}^F = \wedge(F^*)\det(F)$, corresponding to $\psi = \iota(\phi')\chi \in \wedge(F)$, reads

$$(\psi,\phi) = \mathrm{tr}(y^\top x) = \mathrm{tr}(x^\top y) = \sigma(\phi^\top \phi'\chi)_{[0]} = (\iota(\phi^\top)\iota(\phi')\chi)_{[0]} = (\iota(\phi^\top)\psi)_{[0]}$$

which is just the standard pairing between $\wedge(F^*)$ and $\wedge(F)$. $\square$

2.4. Irreducibility of the spinor module. We had encountered special cases of the following result in our discussion of the Clifford algebras for $\mathbb{K} = \mathbb{C}$. See §2, Proposition 2.6.

THEOREM 2.9. *Let* V *be a vector space with split bilinear form* B*, and* $F \subset V$ *a Lagrangian subspace. The spinor module* S_F *is irreducible, and the module map*

$$\varrho\colon\ \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S}_F)$$

is an isomorphism of super algebras. It restricts to an isomorphism

$$\mathrm{Cl}(V)^{\bar{0}} \to \mathrm{End}(\mathsf{S}_F^{\bar{0}}) \oplus \mathrm{End}(\mathsf{S}_F^{\bar{1}}).$$

Hence both $\mathsf{S}_F^{\bar{0}}, \mathsf{S}_F^{\bar{1}}$ *are irreducible modules over* $\mathrm{Cl}(V)^{\bar{0}}$*. These two modules are non-isomorphic.*

PROOF. We may use the model $V = F^* \oplus F$, $\mathsf{S}_F = \wedge(F^*)$. To prove $\mathrm{Cl}(V) \cong \mathrm{End}(\mathsf{S}_F)$, note that both spaces have the same dimension. Hence it suffices to show that ϱ is surjective. That is, we have to show that $\mathrm{End}(\wedge F^*)$ is generated by exterior multiplications and contractions. Suppose first that $\dim F = 1$, and let $e \in F$, $f \in F^*$ be dual generators, so that $B(e,f) = \frac{1}{2}$. Then $\wedge F^*$ has basis $1, f$. In terms of this basis,

$$\epsilon(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \iota(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \epsilon(f)\iota(e) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Together with the identity these form a basis of $\mathrm{End}(\wedge F^*) \cong \mathrm{Mat}_2(\mathbb{K})$, as claimed. The general case follows from the 1-dimensional case, using

$$\mathrm{End}(\wedge(F_1^* \oplus F_2^*)) = \mathrm{End}(\wedge F_1^*) \otimes \mathrm{End}(\wedge F_2^*).$$

This proves $\mathrm{Cl}(V) \cong \mathrm{End}(\mathsf{S}_F)$, which also implies that the spinor module is irreducible. It also gives

$$\mathrm{Cl}(V)^{\bar{0}} \cong \mathrm{End}^{\bar{0}}(\mathsf{S}_F) = \mathrm{End}(\mathsf{S}_F^{\bar{0}}) \oplus \mathrm{End}(\mathsf{S}_F^{\bar{1}}),$$

a direct sum of two irreducible modules. To see that the even and odd part of the spinor module are non-isomorphic modules over $\mathrm{Cl}^{\bar{0}}(V)$, choose bases e_i of F and f^i of F^* such that $B(e_i, f^j) = \frac{1}{2}\delta_i^j$, thus $e_i f^j = \delta_i^j - f^j e_i$. Consider the chirality element (20), written in the 'normal-ordered' form

$$\Gamma = (1 - 2f^1 e_1) \cdots (1 - 2f^m e_m). \tag{35}$$

Since $\varrho(1 - 2f^i e_i) f^I = \pm f^I$, with a $-$ sign if $i \in I$ and a minus sign if $i \notin I$, we find that $\varrho(\Gamma)$ is the parity operator on S_F: it acts as $+1$ on $\mathsf{S}_F^{\bar{0}}$ and as -1 on $\mathsf{S}_F^{\bar{1}}$. In particular, these two representations are non-isomorphic. □

REMARK 2.10. The restrictions of the homomorphism ϱ to $\wedge(F), \wedge(F^*)$ are the extensions of contractions and exterior multiplications as algebra homomorphisms (still denoted ι, ϵ). Using the Proposition, we obtain that the linear map

$$\wedge(F^*) \otimes \wedge(F) \to \mathrm{End}(\wedge(F)), \quad \sum_i \phi_i \otimes \psi_j \mapsto \sum_i \epsilon(\phi_i)\iota(\psi_i^\top) \tag{36}$$

is an isomorphism of super vector spaces. The operators on the right hand side may be thought of as differential operators on the super algebra $\wedge(F^*)$.

2.5. Abstract spinor modules. Theorem 2.9 motivates the following definition.

DEFINITION 2.11. Let V be a vector space with split bilinear form. A *spinor* module over $\mathrm{Cl}(V)$ is a Clifford module S for which the Clifford action

$$\varrho\colon \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$$

is an isomorphism of super algebras. An *ungraded spinor module* is defined as an ungraded Clifford module such that ϱ is an isomorphism of (ordinary) algebras.

We stress that we take Clifford modules, spinor module etc. to be $\mathbb{Z}_2$-graded unless specified otherwise. By Theorem 2.9 the standard spinor module S_F is an example of a spinor module, as is its dual S^F.

THEOREM 2.12. *Let V be a vector space with split bilinear form.*

1. *There is a unique isomorphism class of ungraded spinor modules over* $\mathrm{Cl}(V)$.
2. *There are exactly two isomorphism classes of spinor modules over* $\mathrm{Cl}(V)$, *represented by* $\mathsf{S}_F, \mathsf{S}_F^{\mathrm{op}}$.
3. *A given ungraded spinor module admits exactly two compatible $\mathbb{Z}_2$-gradings. The corresponding parity operator is given by the Clifford action of the chirality element Γ, normalized (up to sign) by the condition $\Gamma^2 = 1$.*

PROOF. Theorem 2.9 shows that as an ungraded algebra, $\mathrm{Cl}(V)$ is isomorphic to a matrix algebra. Hence it admits, up to isomorphism, a *unique* ungraded spinor module, proving (1). The chirality element $\Gamma \in \mathrm{Cl}(V)$ (cf. (35)) satisfies $v\Gamma = -\Gamma v$ for all $v \in V$. Hence $\varrho(v)$ exchanges the ± 1 eigenspaces of $\varrho(\Gamma)$, showing that $\varrho(\Gamma)$ defines a compatible $\mathbb{Z}_2$-grading. Now suppose $\mathsf{S} = \mathsf{S}^{\bar{0}} \oplus \mathsf{S}^{\bar{1}}$ is any compatible $\mathbb{Z}_2$-grading. Since $\varrho(v)$ for $v \neq 0$ exchanges the odd and even summands, they both have dimension $\frac{1}{2} \dim \mathsf{S}$. Hence they are both irreducible under the action of $\mathrm{Cl}(V)^{\bar{0}}$. Since Γ is in the center of $\mathrm{Cl}(V)^{\bar{0}}$, it acts as a scalar on each summand. It follows that $\mathsf{S}^{\bar{0}}$ must be one of the two eigenspaces of $\varrho(\Gamma)$, and $\mathsf{S}^{\bar{1}}$ is the other. This proves (3). Part (2) is immediate from (1) and (3). ☐

The Theorem shows that if S, S' are two spinor modules, then the space

$$\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}, \mathsf{S}')$$

of intertwining operators is a 1-dimensional super vector space.

REMARK 2.13. For $\mathbb{K} = \mathbb{R}$, the choice of $\mathbb{Z}_2$-grading on an ungraded spinor module over S is equivalent to a choice of orientation of V. Indeed, the definition (20) of the chirality element $\Gamma \in \mathrm{Spin}(V)$ shows that the choice of sign of Γ is equivalent to a choice of orientation on V.

As a special case, it follows that the spinor modules defined by two Lagrangian subspaces F, F' are isomorphic, possibly up to parity reversal, where the isomorphism is unique up to a scalar. Recall that $\mathrm{O}(V)$ acts transitively on the set $\mathrm{Lag}(V)$ of Lagrangian subspaces of V. Furthermore, the stabilizer group $\mathrm{O}(V)_F$ of any $F \in \mathrm{Lag}(V)$ is contained in $\mathrm{SO}(V)$.

DEFINITION 2.14. We say that $F, F' \in \mathrm{Lag}(V)$ have *equal parity* if they are related by a transformation $g \in \mathrm{SO}(V)$ and *opposite parity* otherwise.

(For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the relative parity indicates if F, F' are in the same component of $\mathrm{Lag}(V)$.)

PROPOSITION 2.15. *Let $g \in \mathrm{O}(V)$ with $g.F = F'$. Then any lift $x \in \Gamma(V)$ of g determines an isomorphism of Clifford modules $\mathsf{S}_F \to \mathsf{S}_{F'}$. This isomorphism preserves parity if and only if F, F' have the same parity.*

PROOF. Suppose $x \in \Gamma(V)$ lifts g, so that $A_x = g$. Then $F' = A_x(F) = \Pi(x) F x^{-1}$ (as subsets of $\mathrm{Cl}(V)$). Hence

$$\mathrm{Cl}(V) F x^{-1} = \mathrm{Cl}(V) F'.$$

Thus, right multiplication by x^{-1} on $\mathrm{Cl}(V)$ descends to an isomorphism of Clifford modules $\mathsf{S}_F \to \mathsf{S}_{F'}$. Note that this isomorphism preserves parity if and only if x is even, i.e. $g \in \mathrm{SO}(V)$. ☐

Given a spinor module S over $\mathrm{Cl}(V)$, one obtains by restriction a group representation of the Clifford group $\Gamma(V)$ and its subgroup $\mathrm{Pin}(V)$. This is called the *spin representation* of $\Gamma(V)$. The action of $S\Gamma(V)$ preserves the splitting $\mathsf{S} = \mathsf{S}^{\bar{0}} \oplus \mathsf{S}^{\bar{1}}$; the two summands are called the *half-spin representations* of $S\Gamma(V)$ and of its subgroup $\mathrm{Spin}(V)$.

THEOREM 2.16. *The spin representation of* $\Gamma(V)$ *on* S *is irreducible. Similarly, each of the half-spin representations* $\mathsf{S}^{\bar{0}}$ *and* $\mathsf{S}^{\bar{1}}$ *is an irreducible representation of* $S\Gamma(V)$. *The two half-spin representations of* $S\Gamma(V)$ *are non-isomorphic. If* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$ *we can replace* $\Gamma(V)$ *with* $\mathrm{Pin}(V)$ *and* $S\Gamma(V)$ *with* $\mathrm{Spin}(V)$.

PROOF. If a subspace of S is invariant under the action of $\Gamma(V)$, then it is also invariant under the subalgebra generated by $\Gamma(V)$. But $\Gamma(V)$ contains in particular all non-isotropic vectors, and linear combinations of non-isotropic vectors span all of V and hence generate all of $\mathrm{Cl}(V)$. Hence the subalgebra generated by $\Gamma(V)$ is all of $\mathrm{Cl}(V)$, and the irreducibility under $\Gamma(V)$ follows from that under $\mathrm{Cl}(V)$. Similarly, the subalgebra generated by $S\Gamma(V)$ equals $\mathrm{Cl}^{\bar{0}}(V)$, and the irreducibility of the half-spin representations under $S\Gamma(V)$ follows from that under $\mathrm{Cl}^{\bar{0}}(V)$. $\square$

3. Pure spinors

Let $\varrho\colon \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$ be a spinor module. If $\phi \in \mathsf{S}$ is a non-zero spinor, we can consider the space of vectors in V which annihilate ϕ under the Clifford action:

$$F(\phi) = \{v \in V |\ \varrho(v)\phi = 0\}.$$

LEMMA 3.1. *For all non-zero spinors* $\phi \in \mathsf{S}$, *the space* $F(\phi)$ *is an isotropic subspace of* V.

PROOF. If $v_1, v_2 \in F(\phi)$ we have

$$0 = (\varrho(v_1)\varrho(v_2) + \varrho(v_2)\varrho(v_1))\phi = 2B(v_1, v_2)\phi,$$

hence $B(v_1, v_2) = 0$. $\square$

DEFINITION 3.2. A non-zero spinor $\phi \in \mathsf{S}$ is called *pure* if the subspace $F(\phi)$ is Lagrangian.

Consider for instance the standard spinor module $\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F$ defined by a Lagrangian subspace F. Let $\phi_0 \in \mathsf{S}_F$ be the image of $1 \in \mathrm{Cl}(V)$. Then ϕ_0 is a pure spinor, with $F(\phi_0) = F$.

THEOREM 3.3. *The representation of* $\Gamma(V)$ *on a spinor module* S *restricts to a transitive action on the set of pure spinors. The map*

$$\left\{ \begin{array}{c} \textit{pure} \\ \textit{spinors} \end{array} \right\} \to \mathrm{Lag}(V),\ \ \phi \mapsto F(\phi) \tag{37}$$

is a $\Gamma(V)$*-equivariant surjection, with fibers* $\mathbb{K}^\times$. *That is, if* $F(\phi) = F(\phi')$, *then* ϕ, ϕ' *coincide up to a non-zero scalar. All pure spinors* ϕ *are either even or odd. The relative parity of pure spinors* ϕ, ϕ' *is equal to the relative parity of the Lagrangian subspaces* $F(\phi)$, $F(\phi')$.

Proof. For any $x \in \Gamma(V)$, mapping to $g \in \mathrm{O}(V)$,
$$F(\varrho(x)\phi) = xF(\phi)x^{-1} = \Pi(x)F(\phi)x^{-1} = A_x.F(\phi) = g.F(\phi).$$
It follows that for any pure spinor ϕ, the element $\varrho(x)\phi$ is again a pure spinor.

To prove the remaining claims, we work with the standard spinor module S_F defined by a fixed Lagrangian subspace F. Let $\phi_0 \in \mathsf{S}_F$ be the image of $1 \in \mathrm{Cl}(V)$, so that $F(\phi_0) = F$. Suppose ϕ is a pure spinor with $F(\phi) = F$, and consider the standard filtration on S_F. By Proposition 2.6, $\varrho(v)$ for $v \in F$ has filtration degree -1, and $\mathrm{gr}^{-1}(\varrho(v))$ is the operator of contraction by $\mathrm{gr}(\mathsf{S}_F) = \wedge(F^*)$ given by contraction with v. Since $\bigcap_{v \in V} \ker(\iota(v)) = \wedge^0(F^*) = \mathbb{K}$, we conclude that $\bigcap_{v\in V} \ker(\varrho(v)) = \mathsf{S}_F^{(0)} = \mathbb{K}\phi_0$. That is, $F \subseteq F(\phi)$ if and only if ϕ is a scalar multiple of ϕ_0.

Consider now a general pure spinor ϕ. Pick $g \in \mathrm{O}(V)$ with $g.F(\phi) = F$, and choose a lift $x \in \Gamma(V)$ of g. Then $F(\varrho(x)\phi) = g.F(\phi) = F$, so that $\varrho(x)\phi$ is a scalar multiple of ϕ_0. Since $\mathbb{K}^\times \subset \Gamma(V)$, this shows that $\Gamma(V)$ acts transitively on the set of pure spinors. The last statement follows since any element of the Clifford group is either even or odd, thus $\varrho(x)^{-1}\phi_0$ is even or odd depending on the parity of x. $\square$

The following Proposition shows how the choice of a pure spinor identifies S with a spinor module of the form S_F.

Proposition 3.4. *Let S be a spinor module over* $\mathrm{Cl}(V)$.

(i) *For any pure spinor $\phi \in \mathsf{S}$, one has*
$$\{x \in \mathrm{Cl}(V) |\ \varrho(x)\phi = 0\} = \mathrm{Cl}(V)F(\phi).$$
Hence, there is a unique isomorphism of spinor modules $\mathsf{S} \to \mathsf{S}_{F(\phi)}$ taking ϕ to the image of 1 *in* $\mathrm{Cl}(V)/\mathrm{Cl}(V)F(\phi)$. *This identification preserves or reverses the $\mathbb{Z}_2$-grading, depending on the parity of ϕ.*

(ii) *Suppose $F \subset V$ Lagrangian. Then the pure spinors defining F are exactly the non-zero elements of the* pure spinor line
$$l_F = \{\phi \in \mathsf{S} |\ \varrho(v)\phi = 0 \quad \forall v \in F\}.$$
There is a canonical isomorphism,
$$l_F \cong \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}_F, \mathsf{S}).$$

(1) *If S' is another spinor module, and l'_F the pure spinor line for F, then*
$$\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}, \mathsf{S}') = \mathrm{Hom}_{\mathbb{K}}(\mathsf{l}_F, \mathsf{l}'_F)$$
canonically.

Proof. (i) The left ideal $\mathrm{Cl}(V)F(\phi)$ annihilates ϕ, defining a non-zero $\mathrm{Cl}(V)$-equivariant homomorphism $\mathsf{S}_{F(\phi)} = \mathrm{Cl}(V)/\mathrm{Cl}(V)F(\phi) \to \mathsf{S},\ [x] \mapsto \varrho(x)\phi$. Since $\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}_{F(\phi)}, \mathsf{S})$ is 1-dimensional, this map is an isomorphism. In particular, $\varrho(x)\phi = 0$ if and only if $[x] = 0$, i.e. $x \in \mathrm{Cl}(V)F(\phi)$.
(ii) By definition, l_F consists of spinors ϕ with $F \subset F(\phi)$. If ϕ is non-zero this

must be an equality, since $F(\phi)$ is isotropic. This shows that the non-zero elements of l_F are precisely the pure spinors defining F, and (using Theorem 3.3) that $\dim l_F = 1$. The isomorphism $l_F \cong \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}_F, \mathsf{S})$ is defined by the map taking $\phi \in l_F$ to the homomorphism $\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F \to \mathsf{S},\ [x] \mapsto \varrho(x)\phi$. (iii) A $\mathrm{Cl}(V)$-equivariant isomorphism $\mathsf{S} \to \mathsf{S}'$ must restrict to an isomorphism of the pure spinor lines for any Lagrangian subspace F. This defines a non-zero map $\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}, \mathsf{S}') \to \mathrm{Hom}_{\mathbb{K}}(\mathsf{l}_F, \mathsf{l}'_F)$. Since both sides are 1-dimensional, it is an isomorphism. □

Having established these general results, we proceed to give an explicit description of all pure spinors for the standard spinor module S_F, using a Lagrangian complement to F to identify $V = F^* \oplus F$ and $\mathsf{S}_F = \wedge(F^*)$.

PROPOSITION 3.5. *Let $V = F^* \oplus F$. Then any triple (N, χ, ω_N) consisting of a subspace $N \subseteq F$, a volume form $\kappa \in \det(\mathrm{ann}(N))^\times$ on V/N, and a 2-form $\omega_N \in \wedge^2 N^*$ on N, defines a pure spinor*

$$\phi = \exp(-\tilde{\omega}_N)\kappa \in \wedge(F^*).$$

Here $\tilde{\omega}_N \in \wedge^2 F^*$ is an arbitrary extension of ω_N to a 2-form on F. (Note that ϕ does not depend on the choice of this extension). The corresponding Lagrangian subspace is

$$F(\phi) = \{(\mu, v) \in F^* \oplus F|\ v \in N,\ \forall w \in N\colon \langle \mu, w\rangle = \omega_N(v, w)\ \}.$$

Every pure spinor in S_F arises in this way from a unique triple (N, κ, ω_N).

PROOF. We first observe that Lagrangian subspaces $L \subset F^* \oplus F$ are in bijective correspondence with pairs (N, ω_N). Indeed, any such pair defines a subspace of dimension $\dim F$,

$$L = \{(\mu, v) \in F^* \oplus F|\ v \in N,\ \mu|_N = \omega_N(v, \cdot)\}.$$

If $(\mu, v), (mu', v') \in L$ then

$$\langle \mu, v'\rangle + \langle \mu', v\rangle = \omega_N(v, v') + \omega_N(v', v) = 0,$$

hence L is Lagrangian. Conversely, given $L \subset F^* \oplus F$ let $N \subset F$ be its projection, and define ω_N by

$$\omega_N(v, v') = \langle \mu, v'\rangle = -\langle \mu', v\rangle$$

where $(\mu, v), (\mu', v') \in L$ are pre-images of v, v'.

Suppose now that (N, ω_N, κ) are given, and define ϕ as above. It is straightforward to check that elements (μ, v) with $v \in N$ and $\omega|_N = \omega_N(v, \cdot)$ annihilate ϕ. Hence $F(\phi)$ contains all such elements, and equality follows for dimension reasons. Conversely, suppose ϕ is a given pure spinor. Let $N \subset F$ be the projection of $F(\phi) \subset F^* \oplus F$ to F. Then $\mathrm{ann}(N) \subset F(\phi)$. Pick $\kappa \in \det(\mathrm{ann}(N))^\times$. For $v, w \in N$, let $\mu, \nu \in F^*$ such that $(\mu, v), (\nu, w) \in F(\phi)$. Since $F(\phi)$ is isotropic, we have $\langle \mu, w\rangle + \langle \nu, v\rangle = 0$. Hence

$$\omega_N(v, w) = \langle \mu, w\rangle$$

is a well-defined skew-symmetric 2-form on N. Let $\tilde{\omega}_N$ be an arbitrary extension to a 2-form on V. Then $F(\phi)$ has the description given in the Proposition, and hence coincides with $F(e^{-\tilde{\omega}_N}\kappa)$. It follows that ϕ is a non-zero scalar multiple of $e^{-\tilde{\omega}_N}\kappa$, where the scalar can be absorbed in the choice of κ. $\square$

In low dimensions, it is easy to be pure:

PROPOSITION 3.6. *Suppose V is a vector space with split bilinear form, with $\dim V \le 6$, and S a spinor module. Then all non-zero even or odd elements in S are pure spinors.*

PROOF. Consider the case $\dim V = 6$ (the case $\dim V < 6$ is even easier). We may use the model $V = F^* \oplus F$, $\mathsf{S} = \wedge(F^*)$. Suppose $\phi = \phi_{[0]} + \phi_{[2]} \in \mathsf{S}^{\bar{0}}$ is non-zero. If $t := \phi_{[0]} \neq 0$ we have $\phi = t\exp(\phi_{[2]}/t)$, which is a pure spinor by Proposition 3.5. If $\phi_{[0]} = 0$, then $\chi := \phi_{[2]}$ is a non-zero element of $\wedge^2 F^*$. Since $\dim F = 3$, it has a 1-dimensional kernel $N \subset F$, with χ a generator of $\det(\mathrm{ann}(N))$. But this again is a pure spinor by Proposition 3.5.

For non-zero odd spinors $\phi \in \mathsf{S}^{\bar{1}}$, choose a non-isotropic $v \in V$ with $\varrho(v)\phi \neq 0$. Since $\phi' = \varrho(v)\phi \in \mathsf{S}^{\bar{0}}$ is pure, the same is true of $\phi = B(v,v)^{-1}\varrho(v)\phi'$. $\square$

4. The canonical bilinear pairing on spinors

Given a spinor module S, the dual S^* is again a spinor module. The 1-dimensional super vector space

$$K_{\mathsf{S}} := \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}^*, \mathsf{S})$$

is called the *canonical line* for the spinor module. Its parity is even or odd depending on the parity of $\frac{1}{2}\dim V$. The evaluation map defines an isomorphism of Clifford modules,

$$\mathsf{S}^* \otimes K_{\mathsf{S}} \to \mathsf{S}.$$

Note also that $K_{\mathsf{S}^*} = K_{\mathsf{S}}^*$.

By Proposition 3.4, if $F \subset V$ is a Lagrangian subspace and $\mathsf{l}_{\mathsf{S}^*,F}$, $\mathsf{l}_{\mathsf{S},F}$ the corresponding pure spinor lines, we have

$$K_{\mathsf{S}} = \mathrm{Hom}_{\mathbb{K}}(\mathsf{l}_{\mathsf{S}^*,F}, \mathsf{l}_{\mathsf{S},F}) = \mathsf{l}_{\mathsf{S},F} \otimes (\mathsf{l}_{\mathsf{S}^*,F})^*.$$

EXAMPLE 4.1. Let $F \subset V$ be a Lagrangian subspace. We had seen above that the dual of $\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F$ is canonically isomorphic to $\mathsf{S}^F = \mathrm{Cl}(V)\det(F)$. The pure spinor lines l_F for these two spinor modules are

$$\mathsf{l}_{\mathsf{S}^F,F} = \det(F), \qquad \mathsf{l}_{\mathsf{S}_F,F} = \mathbb{K}.$$

Hence

$$K_{\mathsf{S}^F} = \det(F), \quad K_{\mathsf{S}_F} = \det(F^*).$$

In terms of the identifications $\mathsf{S}_F = \wedge(F^*)$, $\mathsf{S}^F = \wedge(F)$ given by the choice of a complementary Lagrangian subspace, the isomorphism

$$K_{\mathsf{S}^F} = \mathrm{Hom}_{\mathrm{Cl}(V)}(\wedge(F), \wedge(F^*)) = \det(F^*)$$

is given by the contraction $\wedge(F) \otimes \det(F^*) \to \wedge(F^*)$. Indeed, given a generator of $\det(F^*)$ the resulting map $\wedge(F) \to \wedge(F^*)$ intertwines $\iota(\mu), \epsilon(v)$ with $\epsilon(\mu), \iota(v)$.

Definition 4.2. The canonical bilinear pairing

$$(\cdot,\cdot)_{\mathsf{S}}\colon \quad \mathsf{S} \otimes \mathsf{S} \to K_{\mathsf{S}}, \quad \phi \otimes \psi \mapsto (\phi, \psi)_{\mathsf{S}}$$

is the isomorphism $\mathsf{S} \otimes \mathsf{S} \to \mathsf{S}^* \otimes \mathsf{S} \otimes K_{\mathsf{S}}$ followed by the duality pairing $\mathsf{S}^* \otimes \mathsf{S} \to \mathbb{K}$.

The pairing $(\cdot,\cdot)_{\mathsf{S}}$ satisfies

$$(\varrho(x)\phi, \psi)_{\mathsf{S}} = (\phi, \varrho(x^\top)\psi)_{\mathsf{S}}, \quad x \in \mathrm{Cl}(V), \tag{38}$$

by a similar property of the pairing between S^* and S (defining the Clifford action on S^*). Restricting to the Clifford group, and replacing ψ with $\varrho(x)\psi$ it follows that

$$(\varrho(x)\phi,\ \varrho(x)\psi)_{\mathsf{S}} = \mathsf{N}(x)\ (\phi, \psi)_{\mathsf{S}}, \quad x \in \Gamma(V) \tag{39}$$

where $\mathsf{N}\colon \Gamma(V) \to \mathbb{K}^\times$ is the norm homomorphism (33). The bilinear form is uniquely determined, up to a non-zero scalar, by its invariance property:

Proposition 4.3. *Suppose $\varrho\colon \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$ is a spinor module, and $(\cdot,\cdot)\colon \mathsf{S} \times \mathsf{S} \to \mathbb{K}$ is a bilinear pairing with the property*

$$(\varrho(x)\phi,\ \varrho(x)\psi)_{\mathsf{S}} = \mathsf{N}(x)\ (\phi, \psi)_{\mathsf{S}}, \quad x \in \Gamma(V).$$

Then $(\cdot,\cdot)$ coincides with the canonical pairing $(\cdot,\cdot)_{\mathsf{S}}$, for some trivialization $K_{\mathsf{S}} \cong \mathbb{K}$.

Proof. The invariance property implies that $(\varrho(x)\phi, \psi) = (\phi, \varrho(x^\top)\psi)$ for all $x \in \Gamma(V)$, hence (by linearity) for all $x \in \mathrm{Cl}(V)$. This shows that the bilinear pairing gives an isomorphism of Clifford modules $\mathsf{S} \to \mathsf{S}^*$. It hence provides a trivialization of $K_{\mathsf{S}} = \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}^*, \mathsf{S})$ identifying $(\cdot,\cdot)$ with the pairing $(\cdot,\cdot)_{\mathsf{S}}$. ☐

Example 4.4. For the Clifford module $\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F$ defined by a Lagrangian subspace F, one has $K_{\mathsf{S}_F} = \det(F^*)$. To explicitly write down the pairing, choose a Lagrangian complement in order to identify $\mathsf{S}_F = \wedge(F^*)$. Then the pairing is given by

$$(\phi, \psi)_{\mathsf{S}} = (\phi^\top \wedge \psi)_{[\mathrm{top}]}. \tag{40}$$

We next turn to the symmetry properties of the bilinear form.

Proposition 4.5. *Let $\dim V = 2m$. The canonical pairing $(\cdot,\cdot)_{\mathsf{S}}$ is*

- *symmetric if $m = 0, 1 \mod 4$,*
- *skew-symmetric if $m = 2, 3 \mod 4$.*

Furthermore, if $m = 0 \mod 4$ (resp. $m = 2 \mod 4$) it restricts to a non-degenerate symmetric (resp. skew-symmetric) form on both $\mathsf{S}^{\bar{0}}$ and $\mathsf{S}^{\bar{1}}$. If m is odd, then the bilinear form vanishes on both $\mathsf{S}^{\bar{0}}$, $\mathsf{S}^{\bar{1}}$, and hence gives a non-degenerate pairing between them.

PROOF. We may use the model $V = F^* \oplus F$, $\mathsf{S} = \wedge(F^*)$. Let $\phi \in \wedge^k(F^*)$, $\psi \in \wedge^{m-k}(F^*)$. Then

$$\begin{aligned}(\psi,\phi)_{\mathsf{S}} &= \psi^\top \wedge \phi\\ &= (-1)^{(m-k)(m-k-1)/2}\psi \wedge \phi\\ &= (-1)^{(m-k)(m-k-1)/2+k(m-k)}\phi \wedge \psi\\ &= (-1)^{(m-k)(m-k-1)/2+k(m-k)+k(k-1)/2}\phi^\top \wedge \psi\\ &= (-1)^{m(m-1)/2}(\phi,\psi)_{\mathsf{S}}.\end{aligned}$$

This gives the symmetry property of the bilinear form. If m is even, then $\mathsf{S}^{\bar{0}}$ and $\mathsf{S}^{\bar{1}}$ are orthogonal under this bilinear form, and hence the bilinear form is non-degenerate on both. $\square$

REMARK 4.6. Suppose $m = 0, 1 \mod 4$, so that $(\cdot,\cdot)_{\mathsf{S}}$ is symmetric. Then

$$(\varrho(v)\phi, \varrho(w)\phi)_{\mathsf{S}} = B(v,w)(\phi,\phi)_{\mathsf{S}}. \tag{41}$$

If $v = w$ this follows from the invariance property (since $\mathsf{N}(v) = v^\top v = B(v,v)$, and in the general by polarization. The identity shows that if $(\phi,\phi)_{\mathsf{S}} \neq 0$ then the null space $F(\phi)$ is trivial. Indeed, for all $v \in F(\phi)$ the identity implies $B(v,w) = 0$ for all w, hence $v = 0$.

THEOREM 4.7 (E. Cartan-Chevalley). *Let S be a spinor module. Let ϕ, $\psi \in \mathsf{S}$ be pure spinors. Then the pairing $(\phi,\psi)_{\mathsf{S}}$ is non-zero if and only if the Lagrangian subspaces $F(\phi)$, $F(\psi)$ are transverse.*

PROOF. We work in the model $V = F^* \oplus F$ and $\mathsf{S} = \wedge F^*$, using the formula (40) for the pairing.
'$\Leftarrow$'. Suppose $F(\phi) \cap F(\psi) = 0$. Choose $A \in \mathrm{O}(V)$ such that A^{-1} takes $F(\phi), F(\psi)$ to F^*, F respectively. Let $x \in \Gamma(V)$ be a lift, i.e. $A_x = A$. Then $\varrho(x)^{-1}\phi$, $\varrho(x)^{-1}\psi$ are pure spinors representing F^*, F, hence they are elements of $\det(F^*)^\times$, $\mathbb{K}^\times$ respectively. By (40) their pairing is non-zero, hence also

$$(\phi,\psi)_{\mathsf{S}} = \mathsf{N}(x)\ (\varrho(x)^{-1}\phi, \varrho(x)^{-1}\psi)_{\mathsf{S}} \neq 0.$$

'$\Rightarrow$'. Suppose $(\phi,\psi)_{\mathsf{S}} \neq 0$. Choose $x \in \Gamma(V)$ with $\psi = \varrho(x).1$. Then

$$0 \neq \mathsf{N}(x)(\phi,\psi)_{\mathsf{S}} = (\varrho(x)^{-1}\phi, 1)_{\mathsf{S}} = (\varrho(x)^{-1}\phi)_{[\mathrm{top}]}.$$

In particular, $\varrho(x)^{-1}\phi$ is not annihilated by any non-zero $\varrho(v)$, $v \in F$. Hence $F(\varrho(x)^{-1}\phi) \cap F = 0$, and consequently $F(\phi) \cap F(\psi) = F(\phi) \cap A_x(F) = 0$. $\square$

REMARK 4.8. More generally, we could also consider two different spinor modules $\mathcal{S}, \mathcal{S}'$. One obtains a pairing

$$(\cdot,\cdot)\colon \mathcal{S}' \otimes \mathcal{S} \cong \mathcal{S}^* \otimes \operatorname{Hom}(\mathcal{S}^*, \mathcal{S}') \otimes \mathcal{S} \to \operatorname{Hom}(\mathcal{S}^*, \mathcal{S}').$$

As before, the Lagrangian subspaces defined by $\phi \in \mathcal{S}, \phi' \in \mathcal{S}'$ are transverse if and only if $(\phi, \phi') \neq 0$.

In particular, Theorem 4.7 shows that pure spinors satisfy $(\phi, \phi)_{\mathsf{S}} = 0$. In dimension 8, the converse is true.

PROPOSITION 4.9 (Chevalley). [**22**, IV.1.1] *Suppose V is a vector space with split bilinear form, with $\dim V = 8$, and S a spinor module. Then a non-zero even or odd spinor $\phi \in \mathsf{S}$ is pure if and only if $(\phi, \phi)_{\mathsf{S}} = 0$. Furthermore, the spinors ϕ with $(\phi, \phi)_{\mathsf{S}} \neq 0$ satisfy $F(\phi) = 0$.*

PROOF. We work in the model $V = F^* \oplus F$, with the spinor module $\mathsf{S}_F = \wedge F^*$. Suppose $\phi \in \mathsf{S}_F$ is an even or odd non-zero spinor with $(\phi, \phi)_{\mathsf{S}} = 0$. We will show that ϕ is pure. Suppose first that ϕ is even. Then

$$0 = (\phi, \phi)_{\mathsf{S}} = 2\phi_{[0]}\phi_{[4]} - \phi_{[2]} \wedge \phi_{[2]}.$$

If $\phi_{[0]} \neq 0$, we may rescale ϕ to arrange $\phi_{[0]} = 1$. The property $\phi_{[4]} = \frac{1}{2}\phi_{[2]} \wedge \phi_{[2]}$ then means $\phi = \exp(\phi_{[2]})$, which is a pure spinor. If $\phi_{[0]} = 0$, the property $\phi_{[2]} \wedge \phi_{[2]} = 0$ tells us that $\phi_{[2]} = \mu^1 \wedge \mu^2$ for suitable $\mu^1, \mu^2 \in F^*$. Let ω be a 2-form such that $\phi_{[2]} \wedge \omega = \phi_{[4]}$, then $\phi = \mu_1 \wedge \mu_2 \wedge \exp(\omega)$ which is again a pure spinor. If ϕ is odd, choose any non-isotropic v. Then $\varrho(v)\phi$ is an even spinor, with $(\varrho(v)\phi, \varrho(v)\phi)_{\mathsf{S}} = B(v, v)(\phi, \phi)_{\mathsf{S}} = 0$. Hence $\varrho(v)\phi$ is pure, and consequently ϕ is pure.

On the other hand, if $(\phi, \phi)_{\mathsf{S}} \neq 0$, and $v \in F(\phi)$, then (41) shows

$$0 = (\varrho(v)\phi, \varrho(w)\phi)_{\mathsf{S}} = B(v, w)(\phi, \phi)_{\mathsf{S}}$$

for all $w \in V$. Hence $B(v, w) = 0$ for all w and therefore $v = 0$. $\square$

5. The character $\chi\colon \Gamma(V)_F \to \mathbb{K}^\times$

Let $F \subset V$ be a Lagrangian subspace. By §1, Proposition 3.4 the group $\mathrm{O}(V)_F$ of orthogonal transformations preserving F is contained in $\mathrm{SO}(V)$, and fits into an exact sequence,

$$1 \to \wedge^2(F) \to \mathrm{O}(V)_F \to \mathrm{GL}(F) \to 1.$$

Let $\Gamma(V)_F \subset S\Gamma(V)$ be the pre-image of $\mathrm{O}(V)_F$ in the Clifford group, and $\mathrm{Spin}(V)_F = \Gamma(V)_F \cap \mathrm{Pin}(V)$. Thus $x \in \Gamma(V)_F$ if and only if A_x preserves F. The action $\varrho(x)$ of such an element on the spinor module S must preserve the pure spinor line l_F. This defines a group homomorphism

$$\chi\colon \Gamma(V)_F \to \mathbb{K}^\times,$$

with $\varrho(x)\phi = \chi(x)\phi$ for all $x \in \Gamma(V)_F,\ \phi \in l_F$. Clearly, this character is independent of the choice of S.

Proposition 5.1. *The character χ satisfies*

$$\chi(x)^2 = \mathsf{N}(x)\ \det(A_x|_F)$$

for all $x \in \Gamma(V)_F$. Hence, the restriction of χ to the group $\mathrm{Spin}(V)_F$ *defines a square root of the function $x \mapsto \det(A_x|_F)$.*

Proof. We work in the model $V = F^* \oplus F$, $\mathsf{S} = \wedge(F^*)$, so that $\chi(x) = \varrho(x).1$. It suffices to check the following two cases: (i) A_x fixes F pointwise, and (ii) A_x preserves both F and F^*.

In case (i) A_x is given by an element $\lambda \in \wedge^2(F)$, and hence $x = t\exp(-\lambda)$ for some non-zero $t \in \mathbb{K}$. We have $\det(A_x|_F) = 1$. The action of x on $1 \in \wedge(F^*)$ is multiplication by t, hence $\chi(x) = t$, while $\mathsf{N}(x) = t^2$. This verifies the formula in case (i).

In case (ii), let $Q = A_x|_F \in \mathrm{GL}(F)$. Then $A_x(\mu, v) = ((Q^{-1})^*\mu, Qv)$.

For all $\nu \in F^*$ we have $x\nu x^{-1} = A_x(\nu) = (Q^{-1})^*\nu$, where we used $\Pi(x) = x$. It follows that

$$x\psi x^{-1} = (Q^{-1})^*\psi$$

for all $\psi \in \wedge(F^*) \subset \mathrm{Cl}(V)$. Take $\psi \in \det(F^*)^\times$, so that $(Q^{-1})^*\psi = \frac{1}{\det Q}\psi$. We obtain:

$$\varrho(x)\psi = \varrho(x\psi)1 = \varrho(x\psi x^{-1})\varrho(x)1 = \frac{\chi(x)}{\det Q}\psi.$$

Pairing with $\varrho(x)1 = \chi(x)$, and using the invariance property of the bilinear form, we find

$$\mathsf{N}(x)\psi = (\varrho(x).1, \varrho(x)\psi)_{\mathsf{S}} = \frac{\chi(x)^2}{\det Q}\psi,$$

hence $\chi(x)^2 = \mathsf{N}(x)\det(Q)$. □

Let $\mathbb{K}_\chi$ denotes the $\Gamma(V)_F$-representation on $\mathbb{K}$, with $x \in \Gamma(V)_F$ acting as multiplication by $\chi(x)$.

Proposition 5.2. *This fiber of the associated line bundle*

$$\Gamma(V) \times_{\Gamma(V)_F} \mathbb{K}_\chi \to \mathrm{Lag}(V)$$

at $L \in \mathrm{Lag}(V)$ is the pure spinor line $l_L \subset \mathsf{S}_F$.

Proof. Since $x \in \Gamma(V)_F$ acts as $\chi(x)$ on $\mathsf{l}_F \subset \mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F$, we have $x = \chi(x) \mod \mathrm{Cl}(V)F$ for all $x \in \Gamma(V)_F$. Thus

$$\chi(x)x^{-1} = 1 \mod \mathrm{Cl}(V)F.$$

Now let $(z,t) \in \Gamma(V) \times \mathbb{K}_\chi$, and put $L = A_z(F)$. The map $\mathrm{Cl}(V) \to \mathrm{Cl}(V)$, $y \mapsto tyz$ takes $\mathrm{Cl}(V)L$ to $\mathrm{Cl}(V)F$, hence it descends to an element of $\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}_L, \mathsf{S}_F) = \mathsf{l}_L$. If $(z', t') = (zx^{-1}, \chi(x)t)$ with $x \in \Gamma(V)_F$, then

$$t'yz' = \chi(x)tyzx^{-1} = tyz \mod \mathrm{Cl}(V)F,$$

thus (z', t') defines the same homomorphism as the element (z,t). □

If the map $\mathrm{Pin}(V) \to \mathrm{O}(V)$ is onto (e.g. if $\mathbb{K} = \mathbb{C}$), the line bundle has a description

$$\mathrm{Pin}(V) \times_{\mathrm{Pin}(V)_F} \mathbb{K}_\chi,$$

where $\mathrm{Pin}(V)_F = \mathrm{Spin}(V)_F$ acts via the character $x \mapsto \chi(x) = \det^{1/2}(A_x|_F)$.

6. Cartan's Triality Principle

If $\dim V = 8$, one has the remarkable phenomenon of *triality*, discovered by E. Cartan [**17**].

The following discussion is based on Chevalley's exposition in [**22**]. Let $\varrho\colon \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$ be a spinor module, and let $\Gamma \in \mathrm{Spin}(V)$ be the chirality element, with the unique normalization for which $\varrho(\Gamma)$ is the parity operator of S. Since $\Gamma, 1$ span the center of the algebra $\mathrm{Cl}^{\bar{0}}(V)$, and since the linear span of $\mathrm{Spin}(V)$ is all of $\mathrm{Cl}^{\bar{0}}(V)$, the center of the group $\mathrm{Spin}(V)$ consists of four elements

$$\mathrm{Cent}(\mathrm{Spin}(V)) = \{1, -1, \Gamma, -\Gamma\}. \tag{42}$$

The two half-spin representations $\mathsf{S}^{\bar{0}},\ \mathsf{S}^{\bar{1}}$ of $\mathrm{Spin}(V)$ are irreducible representations of dimension 8. In addition, one has the 8-dimensional representation on V via $\pi\colon\ \mathrm{Spin}(V) \to \mathrm{SO}(V),\ x \mapsto A_x$. These three representations are all non-isomorphic, and are distinguished by the action of the center (42). Indeed, the central element $-1 \in \mathrm{Spin}(V)$ acts trivially on V since $\pi(-1) = I$, while it acts as $-\mathrm{id}$ in the half-spin representations. The triality principle, Theorem 6.1 below, shows that there is a degree 3 automorphism of $\mathrm{Spin}(V)$ relating the three representations.

Form the direct sum

$$A = V \oplus \mathsf{S}^{\bar{0}} \oplus \mathsf{S}^{\bar{1}}.$$

Since $\dim V = 8$, Proposition 4.5 shows that the canonical bilinear form $(\cdot,\cdot)_{\mathsf{S}}$ is symmetric, and restricts to non-degenerate bilinear forms on $\mathsf{S}^{\bar{0}}, \mathsf{S}^{\bar{1}}$. After choice of a trivialization $K_{\mathsf{S}} \cong \mathbb{K}$, it becomes a scalar-valued symmetric bilinear form; its direct sum with the given bilinear form B on V is a non-degenerate symmetric bilinear form B_A on A. The corresponding orthogonal group is denoted $\mathrm{O}(A)$, as usual. Let $\varrho_A\colon S\Gamma(V) \to \mathrm{Aut}(A)$ be the triagonal action on A. Then $\varrho_A(\mathrm{Spin}(V)) \subset \mathrm{O}(A)$.

THEOREM 6.1 (Triality). *There exists an orthogonal automorphism $J \in \mathrm{O}(A)$ of order 3 such that*

$$J(V) = \mathsf{S}^{\bar{1}}, \quad J(\mathsf{S}^{\bar{1}}) = \mathsf{S}^{\bar{0}}, \quad J(\mathsf{S}^{\bar{0}}) = V \tag{43}$$

Furthermore, there is a group automorphism $j \in \mathrm{Aut}(\mathrm{Spin}(V))$ of order 3 such that

$$J \circ \varrho_A(x) = \varrho_A(j(x)) \circ J \tag{44}$$

for all $x \in \mathrm{Spin}(V)$. *One hence obtains a commutative diagram, for* $x \in \Gamma(V)$,

$$\begin{array}{ccccc} V & \xrightarrow[J]{} & \mathsf{S}^{\bar 1} & \xrightarrow[J]{} & \mathsf{S}^{\bar 0} \\ \big\downarrow{\scriptstyle \varrho(x)} & & \big\downarrow{\scriptstyle \varrho(j(x))} & & \big\downarrow{\scriptstyle \varrho(j(j(x)))} \\ V & \xrightarrow[J]{} & \mathsf{S}^{\bar 1} & \xrightarrow[J]{} & \mathsf{S}^{\bar 0} \end{array}$$

A key ingredient in the proof is the following cubic form on A,

$$C_A\colon A \to \mathbb{K},\ \xi = (v, \phi^{\bar 0}, \phi^{\bar 1}) \mapsto (\varrho(v)\phi^{\bar 0}, \phi^{\bar 1})_{\mathsf{S}}, \tag{45}$$

Lemma 6.2. *The cubic form* C_A *satisfies* $C_A(\varrho_A(x)\xi) = \mathsf{N}(x)C_A(\xi)$ *for all* $x \in S\Gamma(V)$. *Hence,* $\varrho_A(x)$, $x \in S\Gamma(V)$ *preserves* C_A *precisely if* $x \in \mathrm{Spin}(V)$.

Proof. For $\xi = (v, \phi^{\bar 0}, \phi^{\bar 1})$,

$$\begin{aligned} C_A(\varrho_A(x)\xi) &= (\varrho(A_x(v))\varrho(x)\phi^{\bar 0}, \varrho(x)\phi^{\bar 1})_{\mathsf{S}} \\ &= (\varrho(x)\varrho(v)\phi^{\bar 0}, \varrho(x)\phi^{\bar 1})_{\mathsf{S}} \\ &= \mathsf{N}(x)(\varrho(v)\phi^{\bar 0}, \phi^{\bar 1})_{\mathsf{S}} = \mathsf{N}(x)C_A(\xi). \end{aligned}$$

∎

We will construct the triality automorphism J in such way that it also preserves C_A.

Lemma 6.3. *Any* $f \in \mathrm{O}(A)$ *preserving each of the subspaces* $V, \mathsf{S}^{\bar 0}, \mathsf{S}^{\bar 1}$ *and preserving the cubic form* C_A *is of the form* $\varrho_A(x)$, *for a unique* $x \in \mathrm{Spin}(V)$.

Proof. For all $\xi = (v, \phi^{\bar 0}, \phi^{\bar 1})$ we find

$$(\varrho(f(v))f(\phi^{\bar 0}), f(\phi^{\bar 1}))_{\mathsf{S}} = (\varrho(v)\phi^{\bar 0}, \phi^{\bar 1})_{\mathsf{S}} = (f(\varrho(v)\phi^{\bar 0}), f(\phi^{\bar 1}))_{\mathsf{S}}$$

where the first equality used invariance of C_A and the second equality the invariance of B_A. Consequently

$$f(\varrho(v)\phi) = \varrho(f(v))f(\phi)$$

for all even spinors ϕ. On the other hand, any odd spinor can be written $\psi = \varrho(v)\phi$, hence the identity gives $f(\psi) = \varrho(f(v))f(\varrho(v)^{-1}\psi) = \varrho(f(v))B(v,v)^{-1}f(\varrho(v)\psi) = \varrho(f(v))^{-1}f(\varrho(v)\psi)$. That is, $f(\varrho(v)\psi) = \varrho(f(v))f(\psi)$ for all odd spinors. This shows

$$\varrho(f(v)) = (f|_{\mathsf{S}}) \circ \varrho(v) \circ (f|_{\mathsf{S}})^{-1}. \tag{46}$$

Since $f|_{\mathsf{S}}$ is an even endomorphism of S, it is of the form $\varrho(x)$ for some $x \in \mathrm{Cl}^{\bar 0}(V)$. Equation (46) shows that $\varrho(f(v)) = \varrho(xvx^{-1}) = \varrho(A_x(v))$, hence $f(v) = A_x(v)$ and in particular $x \in S\Gamma(V)$. We have shown that $f = \varrho_A(x)$. Using again the invariance of C_A and the previous Lemma, we obtain $\mathsf{N}(x) = 1$, so that $x \in \mathrm{Spin}(V)$. ∎

Proof of Theorem 6.1. Pick $n \in V$ and $q \in \mathsf{S}^{\bar{0}}$ with $B(n,n) = 1$, $(q,q)_{\mathsf{S}} = 1$. Let R_n and R_q be the corresponding reflections in $V, \mathsf{S}^{\bar{0}}$. The map

$$V \to \mathsf{S}^{\bar{1}} \colon v \mapsto \varrho(v)q$$

is an isometry (see Remark 4.6); let $T_q \colon \mathsf{S}^{\bar{1}} \to V$ be the inverse map. Define orthogonal involutions $\mu, \tau \in \mathrm{O}(A)$ by

$$\begin{aligned} \mu(v, \phi^{\bar{0}}, \phi^{\bar{1}}) &= (R_n(v),\ \varrho(n)\phi^{\bar{1}},\ \varrho(n)\phi^{\bar{0}}), \\ \tau(v, \phi^{\bar{0}}, \phi^{\bar{1}}) &= (T_q(\phi^{\bar{1}}), R_q(\phi^{\bar{0}}), \varrho(v)q). \end{aligned}$$

Note that μ preserves V and exchanges the spaces $\mathsf{S}^{\bar{0}}, \mathsf{S}^{\bar{1}}$, while τ preserves $\mathsf{S}^{\bar{0}}$ and exchanges the spaces $V, \mathsf{S}^{\bar{1}}$. Hence, the composition

$$J = \tau \circ \mu \in \mathrm{O}(A)$$

satisfies (43). Let us verify that $J^3 = \mathrm{id}_A$. It suffices to check on elements $v \in V \subset A$. We have:

$$\begin{aligned} \mu(v) &= R_n(v) = -nvn, \\ \tau\mu(v) &= -\varrho(nvn)q \\ \mu\tau\mu(v) &= -\varrho(vn)q \\ \tau\mu\tau\mu(v) &= -\varrho(vn)q + 2(\varrho(vn)q, q)_{\mathsf{S}}\, q \\ &= -\varrho(vn)q + 2B(v,n)q \\ &= \varrho(nv)q \\ \mu\tau\mu\tau\mu(v) &= \varrho(v)q \\ \tau\mu\tau\mu\tau\mu(v) &= v. \end{aligned}$$

Hence $J^3 v = v$ as claimed. We next show that the cubic form C_A changes sign under μ, τ, and hence is invariant under J. For $\xi = (v, \phi^{\bar{0}}, \phi^{\bar{1}})$, we have

$$C_A(\mu(\xi)) = (\varrho(R_n(v)n)\phi^{\bar{1}}, \varrho(n)\phi^{\bar{0}})_{\mathsf{S}} = -(\varrho(v)\phi^{\bar{1}}, \phi^{\bar{0}})_{\mathsf{S}} = -C_A(\xi).$$

The computation for τ is a bit more involved. Let $w = T_q(\phi^{\bar{1}})$, so that $\varrho(w)q = \phi^{\bar{1}}$. Then

$$\begin{aligned} C_A(\tau(\xi)) &= (\varrho(w)R_q(\phi^{\bar{0}}), \varrho(v)q)_{\mathsf{S}} \\ &= (R_q(\phi^{\bar{0}}), \varrho(wv)q)_{\mathsf{S}} \\ &= (\phi^{\bar{0}}, \varrho(wv)q)_{\mathsf{S}} - 2(\phi^{\bar{0}}, q)_{\mathsf{S}}(q, \varrho(wv)q)_{\mathsf{S}} \\ &= 2B(v,w)(\phi^{\bar{0}}, q)_{\mathsf{S}} - (\phi^{\bar{0}}, \varrho(v)\phi^{\bar{1}})_{\mathsf{S}} - 2(\phi^{\bar{0}}, q)_{\mathsf{S}}(\varrho(w)q, \varrho(v)q)_{\mathsf{S}} \\ &= -(\phi^{\bar{0}}, \varrho(v)\phi^{\bar{1}})_{\mathsf{S}} \\ &= -(\varrho(v)\phi^{\bar{0}}, \phi^{\bar{1}})_{\mathsf{S}} = -C_A(\xi). \end{aligned}$$

Hence $C_A(J(\xi)) = C_A(\xi)$. Suppose now that $x \in \mathrm{Spin}(V)$. Then

$$J \circ \varrho_A(x) \circ J^{-1}$$

preserves B_A, C_A and the three subspaces $V, \mathsf{S}^{\bar{0}}, \mathsf{S}^{\bar{1}}$. By Lemma 6.3 we may write this composition as $\varrho_A(j(x))$ for a unique element $j(x) \in \mathrm{Spin}(V)$. Using the uniqueness part of Lemma 6.3 we find $j(x_1)j(x_2) = j(x_1x_2)$ and $j(j(j(x))) = x$. $\square$

Remark 6.4. The theory described here carries much further. Using polarization, the cubic form C_A defines a symmetric trilinear form $T_A \in S^3(A^*)$, with $T_A(\xi,\xi,\xi) = C_A(\xi)$. This form defines a 'triality pairing': That is, $T_A(\xi_1,\xi_2,\xi_3)$ is a non-degenerate bilinear form in ξ_1,ξ_2 for arbitrary fixed non-zero ξ_3. In turn, this triality can be used to construct an interesting non-associative product on V, making V into the algebra of *octonions*. A beautiful discussion of this theory may be found in the paper [**9**] by Baez.

7. The Clifford algebra $\mathbb{C}l(V)$

In this Section, we denote by V a vector space over $\mathbb{K} = \mathbb{R}$, with a *positive definite* symmetric bilinear form B. The complexification of the Clifford algebra of V coincides with the Clifford algebra of the complexification of V, and will be denote $\mathbb{C}l(V)$. It carries the additional structure of an involution, coming from the complex conjugation operation on $V^{\mathbb{C}}$, and one can consider *unitary* Clifford modules compatible with this involution. In this Section, we will develop the theory of such unitary Clifford modules, and present a number of applications to the theory of compact Lie groups.

7.1. The Clifford algebra $\mathbb{C}l(V)$. Let $V^{\mathbb{C}}$ be the complexification of V. For $v \in V^{\mathbb{C}}$ we denote by $\overline{v}$ its complex conjugate. The Hermitian inner product of $V^{\mathbb{C}}$ will be denoted $\langle \cdot, \cdot \rangle$, while the extension of B to a complex bilinear form will still be denoted B. Thus $\langle v, w \rangle = B(\overline{v}, w)$ for $v, w \in V^{\mathbb{C}}$. We put

$$\mathbb{C}l(V) := \mathrm{Cl}(V^{\mathbb{C}}) = \mathrm{Cl}(V)^{\mathbb{C}}.$$

The complex conjugation mapping $v \mapsto \overline{v}$ on $V^{\mathbb{C}}$ extends to an conjugate linear algebra automorphism $x \mapsto \overline{x}$ of the complex Clifford algebra $\mathbb{C}l(V)$. Define a conjugate linear anti-automorphism

$$x^* = \overline{x}^\top.$$

Thus $(xy)^* = y^*x^*$ and $(ux)^* = \overline{u}x^*$ for $u \in \mathbb{C}$.

Definition 7.1. A unitary Clifford module over $\mathbb{C}l(V)$ is a Hermitian super vector space E together with a morphism of super $*$-algebras $\mathbb{C}l(V) \to \mathrm{End}(E)$.

Thus, for a unitary Clifford module the action map ϱ satisfies $\varrho(x^*) = \varrho(x)^*$ for all $x \in \mathbb{C}l(V)$. Equivalently, the elements of $v \in V \subset \mathbb{C}l(V)$ act as self-adjoint operators. Note that for a unitary Clifford module, the representations of $\mathrm{Spin}(V)$, $\mathrm{Pin}(V)$ preserve the Hermitian inner product. They are thus unitary representations.

EXAMPLE 7.2. The Clifford algebra $\mathbb{C}l(V)$ itself carries a Hermitian inner product, $\langle x,\, y\rangle = \mathrm{tr}(x^*y)$. Let $\varrho\colon \mathbb{C}l(V) \to \mathrm{End}(\mathbb{C}l(V))$ be the action by left multiplication. For $v \in V \subset V^{\mathbb{C}}$ we have $v^* = v$, hence $\langle x,\, vy\rangle = \langle vx,\, y\rangle$ for all $x \in \mathbb{C}l(V)$. This shows that $\mathbb{C}l(V)$ is a unitary $\mathbb{C}l(V)$-module. Arguing as in the proof of §2 Proposition 2.14, we see that the quantization map intertwines the Hermitian inner product on $\wedge V^{\mathbb{C}}$, given by $\langle v_1\wedge\cdots\wedge v_k,\, w_1\wedge\cdots\wedge w_k\rangle = \det\langle v_i, w_j\rangle$, with the Hermitian inner product $\langle x,\, y\rangle = \mathrm{tr}(x^*y)$ on $\mathbb{C}l(V)$.

Since $\mathbb{C}l(V)$ has a faithful unitary representation, such as in the previous example, it follows that $\mathbb{C}l(V)$ is a C^*-algebra. That is, it carries a unique norm $||\cdot||$ relative to which it is a Banach algebra, and such that the C^*-identity $||x^*x|| = ||x||^2$ is satisfied. This norm is equal to the operator norm in any such presentation, and is explcitly given in terms of the trace by the formula

$$||a|| = \lim_{n\to\infty}\Big(\mathrm{tr}(a^*a)^n\Big)^{\frac{1}{2n}}.$$

Note that this C^*-norm is different from the Hilbert space norm $\mathrm{tr}(a^*a)^{1/2}$.

7.2. The groups $\mathrm{Spin}_c(V)$ and $\mathrm{Pin}_c(V)$. Suppose $x \in \Gamma(V^{\mathbb{C}})$, defining a complex transformation $A_x(v) = (-1)^{|x|}xvx^{-1} \in \mathrm{O}(V^{\mathbb{C}})$ as before.

LEMMA 7.3. *The element $x \in \Gamma(V^{\mathbb{C}})$ satisfies $A_x(v)^* = A_x(v^*)$ for all $v \in V^{\mathbb{C}}$, if and only if x^*x is a positive real number.*

PROOF. For all $x \in \Gamma(V^{\mathbb{C}})$ and all $v \in V^{\mathbb{C}}$, we have

$$A_x(v)^* = (-1)^{|x|}(x^{-1})^*v^*x^* = A_{(x^{-1})^*}(v^*).$$

This coincides with $A_x(v^*)$ for all v if and only if $x = \lambda(x^{-1})^*$ for some $\lambda \in \mathbb{C}^\times$, i.e. if and only if $x^*x \in \mathbb{C}^\times$. Since x^*x is a positive element, this condition is equivalent to $x^*x \in \mathbb{R}_{>0}$. ☐

DEFINITION 7.4. We define

$$\Gamma_c(V) = \{x \in \Gamma(V^{\mathbb{C}})|\ x^*x \in \mathbb{R}_{>0}\}$$
$$\mathrm{Pin}_c(V) = \{x \in \Gamma(V^{\mathbb{C}})|\ x^*x = 1\}$$
$$\mathrm{Spin}_c(V) = \mathrm{Pin}_c(V) \cap S\Gamma(V^{\mathbb{C}}).$$

If $V = \mathbb{R}^n$ with the standard bilinear form, we write $\Gamma_c(n)$, $\mathrm{Pin}_c(n)$, $\mathrm{Spin}_c(n)$.

By definition, an element x of the Clifford group lies in $\Gamma_c(V)$ if and only if the automorphism $A_x \in \mathrm{O}(V^{\mathbb{C}})$ preserves the real subspace V. That is, $\Gamma_c(V) \subset \Gamma(V^{\mathbb{C}})$ is the inverse image of $\mathrm{O}(V) \subset \mathrm{O}(V^{\mathbb{C}})$. The exact sequence for $\Gamma(V^{\mathbb{C}})$ restricts to an exact sequences,

$$1 \to \mathbb{C}^\times \to \Gamma_c(V) \to \mathrm{O}(V) \to 1$$
$$1 \to \mathrm{U}(1) \to \mathrm{Pin}_c(V) \to \mathrm{O}(V) \to 1,$$
$$1 \to \mathrm{U}(1) \to \mathrm{Spin}_c(V) \to \mathrm{SO}(V) \to 1,$$

where we have used $\mathbb{C}^\times \cap \mathrm{Pin}_c(V) = \mathbb{C}^\times \cap \mathrm{Spin}_c(V) = \mathrm{U}(1)$.

One can also directly define $\mathrm{Pin}_c(V)$, $\mathrm{Spin}_c(V)$ as the subgroups of $\Gamma(V^\mathbb{C})$ generated by $\mathrm{Pin}(V)$, $\mathrm{Spin}(V)$ together with $\mathrm{U}(1)$. That is, $\mathrm{Spin}_c(V)$ is the quotient of $\mathrm{Spin}(V) \times \mathrm{U}(1)$ by the relation

$$(x, e^{\sqrt{-1}\psi}) \sim (-x, -e^{\sqrt{-1}\psi})$$

and similarly for $\mathrm{Pin}_c(V)$. A third viewpoint towards these groups, using the spinor module, is described in Section 7.3 below. The norm homomorphism for $\Gamma(V^\mathbb{C})$ restricts to a group homomorphism,

$$\mathsf{N}\colon \Gamma_c(V) \to \mathbb{C}^\times,\ x \mapsto x^\top x.$$

On the subgroup $\mathrm{Pin}_c(V)$ this may be written $\mathsf{N}(x) = \overline{x}^{-1}x$, which evidently takes values in $\mathrm{U}(1)$.

Together with the map to $\mathrm{O}(V)$ this defines exact sequences,

$$\begin{aligned}
&1 \to \mathbb{Z}_2 \to \Gamma_c(V) \to \mathrm{O}(V) \times \mathbb{C}^\times \to 1,\\
&1 \to \mathbb{Z}_2 \to \mathrm{Pin}_c(V) \to \mathrm{O}(V) \times \mathrm{U}(1) \to 1,\\
&1 \to \mathbb{Z}_2 \to \mathrm{Spin}_c(V) \to \mathrm{SO}(V) \times \mathrm{U}(1) \to 1
\end{aligned}$$

One of the motivations for introducing the group $\mathrm{Spin}_c(V)$ is the following lifting property. Suppose J is an orthogonal complex structure on V, that is, $J \in \mathrm{O}(V)$ and $J^2 = -I$. Such a J exists if and only if $n = \dim V$ is even, and turns V into a vector space over $\mathbb{C}$, with scalar multiplication

$$(a + \sqrt{-1}b)x = ax + bJx.$$

Let $U_J(V) \subset \mathrm{SO}(V)$ be the corresponding unitary group (i.e. the elements of $\mathrm{SO}(V)$ commuting with J).

THEOREM 7.5. *The inclusion $U_J(V) \hookrightarrow \mathrm{SO}(V)$ admits a unique lift to a group homomorphism $U_J(V) \hookrightarrow \mathrm{Spin}_c(V)$, in such a way that its composite with the map $\mathsf{N}\colon \mathrm{Spin}_c(V) \to \mathrm{U}(1)$ is the complex determinant $\mathrm{U}_J(V) \to \mathrm{U}(1),\ A \mapsto \det_J(A)$.*

PROOF. We have to show that the map

$$U_J(V) \to \mathrm{SO}(V) \times \mathrm{U}(1),\ A \mapsto (A,\ \det_J(A))$$

lifts to the double cover. Since $U_J(V)$ is connected, if such a lift exists then it is unique. To prove existence, it suffices to check that any loop representing the generator of $\pi_1(\mathrm{U}_J(V)) \cong \mathbb{Z}$ lifts to a loop in $\mathrm{Spin}_c(V)$. The inclusion of any non-zero J-invariant subspace $V' \subset V$ induces an isomorphism of the fundamental groups of the unitary groups. It is hence sufficient to check for the case $V = \mathbb{R}^2$, with J the standard complex structure $Je_1 = e_2,\ Je_2 = -e_1$. Our task is to lift the map

$$\mathrm{U}(1) \to \mathrm{SO}(2) \times \mathrm{U}(1),\ e^{\sqrt{-1}\theta} \mapsto (R(\theta), e^{\sqrt{-1}\theta})$$

to the double cover, $\mathrm{Spin}_c(\mathbb{R}^2)$. This lift is explicitly given by the following modification of Example 1.11,

$$x(\theta) = e^{\sqrt{-1}\theta/2}(\cos(\theta/2) + \sin(\theta/2)\, e_1e_2) \in \mathrm{Spin}_c(\mathbb{R}^2).$$

Indeed, $x(\theta + 2\pi) = x(\theta)$ and $\mathsf{N}(x(\theta)) = e^{\sqrt{-1}\theta}$. □

REMARK 7.6. The two possible square roots of $\det_J(A)$ for $A \in \mathrm{U}_J(V)$ define a double cover,

$$\tilde{U}_J(V) = \{(A, z) \in \mathrm{U}_J(V) \times \mathbb{C}^\times |\; z^2 = \det_J(A)\}.$$

While the inclusion $\mathrm{U}_J(V) \hookrightarrow \mathrm{SO}(V)$ does not live to the Spin group, the above proof shows that there exists a lift $\tilde{U}_J(V) \to \mathrm{Spin}(V)$ for this double cover. Equivalently, $\tilde{U}_J(V)$ is identified with the pull-back of the spin double cover.

7.3. Spinor modules over $\mathbb{C}l(V)$. We will now discuss special features of spinor modules over the complex Clifford algebra $\mathbb{C}l(V)$, for an even-dimensional real Euclidean vector space V.

The first point we wish to stress is that, similar to Remark 2.13, *the choice of a compatible $\mathbb{Z}_2$-grading on a spinor module S is equivalent to the choice of orientation on V.* Indeed, let $e_1, \ldots, e_{2m}$ be an oriented orthonormal basis of V, where $\dim V = 2m$. Then the chirality element is

$$\Gamma = (\sqrt{-1})^m e_1 \cdots e_{2m} \in \mathrm{Spin}_c(V).$$

The normalization of Γ is such that $\Gamma^2 = 1$. Changing the orientation changes the sign of Γ, and hence changes the parity operator $\varrho(\Gamma)$.

If the spinor module is of the form S_F for a Lagrangian subspace $F \in \mathrm{Lag}(V^{\mathbb{C}})$, we also have the orientation defined by the complex structure J corresponding to F. (See §1, Section 7.) These two orientations agree:

PROPOSITION 7.7. *Let F be a Lagrangian subspace of $V^{\mathbb{C}}$, and J the corresponding orthogonal complex structure having F as its $+\sqrt{-1}$ eigenspace. Then the orientation on V defined by J coincides with that defined by the $\mathbb{Z}_2$-grading on S_F.*

PROOF. The orientation defined by J is given by the volume element $e_1 \wedge \cdots \wedge e_n$, where e_i is an orthonormal such that $Je_{2j-1} = e_{2j}$. The Lagrangian subspace F is spanned by the orthonormal (for teh Hermitian metric) vectors

$$E_j = \frac{1}{\sqrt{2}}(e_{2j-1} - \sqrt{-1}e_{2j}).$$

We claim that the chirality element given by the basis e_i acts as $+1$ on $\mathsf{S}_F^{\bar{0}}$ and as -1 on $\mathsf{S}_F^{\bar{1}}$. We have

$$E_j\overline{E}_j = \frac{1}{2}(e_{2j-1} - \sqrt{-1}e_{2j})(e_{2j-1} + \sqrt{-1}e_{2j}) = \sqrt{-1}e_{2j-1}e_{2j} + 1,$$

hence

$$\Gamma = (E_1\overline{E}_1 - 1) \cdots (E_m\overline{E}_m - 1).$$

For $I = \{i_1, \ldots, i_k\}$ let $\overline{E}_I = \overline{E}_{i_1} \wedge \cdots \wedge \overline{E}_{i_k} \in \wedge \overline{F} \cong \mathsf{S}_F$. The operator $\varrho(E_i\overline{E}_i - 1)$ acts as 0 on $\overline{E}_I$ if $i \notin I$, and as -1 if $i \in I$. Hence $\varrho(\Gamma)$ acts on $\overline{E}_I$ as $(-1)^k$, proving the claim. $\square$

As a special case of unitary Clifford module, we have unitary spinor modules. These are Clifford modules S with the property that $\varrho\colon \mathbb{C}l(V) \to \mathrm{End}(\mathsf{S})$ is an isomorphism of super $*$-algebras. Equivalently, the $\mathbb{C}l(V)$-action on S is irreducible.

Proposition 7.8. *Any spinor module* S *admits a Hermitian metric, unique up to positive scalar, for which it becomes a unitary spinor module.*

Proof. Let $F \in \mathrm{Lag}(V^{\mathbb{C}})$. Then S_F is a unitary spinor module, and the choice of an isomorphism $\mathsf{S} \cong \mathsf{S}_F$ determines a Hermitian metric on S. Conversely, this Hermitian metric is uniquely determined by its restriction to the pure spinor line l_F, since $\varrho(\mathrm{Cl}(V))l_F = \mathsf{S}$. $\square$

For any two unitary spinor modules S, S', the space $\mathrm{Hom}_{\mathbb{C}l}(\mathsf{S}, \mathsf{S}')$ of intertwining operators inherits a Hermitian metric from the full space of homomorphisms $\mathrm{Hom}(\mathsf{S}, \mathsf{S}')$, and the map

$$\mathsf{S} \otimes \mathrm{Hom}_{\mathbb{C}l}(\mathsf{S}, \mathsf{S}') \to \mathsf{S}'$$

is an isomorphism of unitary Clifford modules. Taking $\mathsf{S}' = \mathsf{S}$, we have $\mathrm{Hom}_{\mathbb{C}l}(\mathsf{S}, \mathsf{S}) = \mathbb{C}$ and the group of $\mathbb{C}l(V)$-equivariant unitary automorphisms of S is the group $\mathrm{U}(1) \subset \mathbb{C}$.

Using unitary spinor modules, one obtains another characterization of the groups $\mathrm{Pin}_c(V)$ and $\mathrm{Spin}_c(V)$. Suppose $\dim V$ is even, and pick a unitary spinor module S. An even or odd element $U \in \mathrm{U}(\mathsf{S})$ *implements* $A \in \mathrm{O}(V)$ if

$$\varrho(A(v)) = \det(A)\ U \circ \varrho(v) \circ U^{-1}$$

for all $v \in V$, with $\det(A) = \pm 1$ depending on the parity of U.

Proposition 7.9. *Suppose* $\dim V$ *is even. For any unitary spinor module* S*, the map* $\mathrm{Pin}_c(V) \to \mathrm{U}(\mathsf{S})$ *is injective, with image the group of implementers of orthogonal transformations of* V*. Similarly* $\mathrm{Spin}_c(V)$ *is isomorphic to the group of implementers of special orthogonal transformations.*

Proof. Let $A \in \mathrm{O}(V)$ be given. If $x \in \mathrm{Pin}_c(V)$ is a lifts A, then $\det(A)\, xvx^{-1} = A(v)$ for all $v \in V$, and consequently $U = \varrho(x)$ satisfies $U\varrho(v)U^{-1} = \det(A)\varrho(A(v))$. Conversely, suppose U is a unitary element of parity $\det(A) = \pm 1$, implementing A. Let x be the unique element in $\mathrm{Cl}(V)$ with $\varrho(x) = U$. Thus $\det(A) = (-1)^{|x|}$, and $\varrho((-1)^{|x|}xvx^{-1}) = \varrho(A(v))$, hence $(-1)^{|x|}xvx^{-1} = A(v)$ since ϱ is faithful. Similarly $N(x) = x^*x \in \mathrm{U}(1)$. $\square$

Thus, $\mathrm{Pin}_c(V)$ and $\mathrm{Spin}_c(V)$ are realized as unitary implementers in S, of orthogonal and special orthogonal transformations, respectively.

7.4. Classification of irreducible $\mathbb{C}l(V)$-modules. Let V be a Euclidean vector space. We have seen that if $\dim V$ is even, there are two isomorphism classes of irreducible $\mathbb{C}l(V)$-modules, related by parity reversal. We will now extend this discussion to include the case of $\dim V$ odd. Recall once again that by default, we take Clifford modules to be equipped with a $\mathbb{Z}_2$-grading. Such a module is irreducible if there is no non-trivial invariant $\mathbb{Z}_2$-*graded* subspace. As we will see, the classification of such $\mathbb{Z}_2$-graded Clifford modules is in a sense 'opposite' to the classification of irreducible ungraded Clifford modules.

The orientation on V determines the chirality operator

$$\Gamma = (\sqrt{-1})^{n(n-1)/2} e_1 \cdots e_n \in \mathrm{Pin}_c(V)$$

where $n = \dim V$ and $e_1, \ldots e_n$ is an oriented orthonormal basis; it satisfies $\Gamma^2 = 1$. For n odd, the element Γ is odd, and it is an element of the (ordinary) center of $\mathbb{C}l(V)$. For n even, the element Γ is even, and it lies in the super-center of $\mathbb{C}l(V)$. There is a canonical isomorphism of (ordinary) algebras

$$\mathbb{C}l(V) \to \mathbb{C}l^{\bar{0}}(V \oplus \mathbb{R}), \tag{47}$$

determined (using the universal property) by the map on generators $v \mapsto \sqrt{-1}ve$, where e is the standard basis vector for the $\mathbb{R}$ summand. Since $(\sqrt{-1}ve)^* = -\sqrt{-1}ev^* = \sqrt{-1}v^*e$ this map is a $*$-isomorphism. If n is odd, the isomorphism (47) takes the chirality element of $\mathbb{C}l(V)$ to the chirality element for $\mathbb{C}l(V \oplus \mathbb{R})$, up to a sign.

THEOREM 7.10. *Let V be a Euclidean vector space of dimension n, and $\mathbb{C}l(V)$ its complexified Clifford algebra.*

i *Suppose n is even. Then there are:*
  - *two isomorphism classes of irreducible $\mathbb{Z}_2$-graded $\mathbb{C}l(V)$-modules,*
  - *a unique isomorphism class of irreducible ungraded $\mathbb{C}l(V)$-modules,*
  - *two isomorphism classes of irreducible $\mathbb{C}l^{\bar{0}}(V)$-modules.*

ii *Suppose n is odd. Then there are:*
  - *a unique isomorphism class of irreducible $\mathbb{Z}_2$-graded $\mathbb{C}l(V)$-modules,*
  - *two isomorphism classes of irreducible ungraded $\mathbb{C}l(V)$-modules,*
  - *a unique isomorphism class of irreducible $\mathbb{C}l^{\bar{0}}(V)$-modules.*

Note that an irreducible $\mathbb{Z}_2$-graded module may be reducible as an ungraded module: There may be invariant subspaces which are not $\mathbb{Z}_2$-graded subspaces.

PROOF. We may assume $V = \mathbb{R}^n$, and let $\Gamma_n \in \mathbb{C}l(n)$ be the chirality element for the standard orientation. Note also that the third item in (i),(ii) is equivalent to the second item in (ii),(i) since $\mathbb{C}l(n-1) \cong \mathbb{C}l^0(n)$.

(i) Suppose n is even. We denote by S_n a spinor module of $\mathbb{C}l(n)$, with $\mathbb{Z}_2$-grading given by the orientation of $\mathbb{R}^n$. By the results of Section 2.5,

S_n represents the unique isomorphism class of ungraded irreducible $\mathbb{C}l(n)$-modules, while $\mathsf{S}_n, \mathsf{S}_n^{\mathrm{op}}$ represent the two isomorphism classes if irreducible $\mathbb{Z}_2$-graded spinor modules. (The latter are distinguished by the action of Γ_n.)

(ii) Suppose n is odd. Then

$$\mathbb{C}l(n) \cong \mathbb{C}l^{\bar{0}}(n+1) \cong \mathrm{End}^{\bar{0}}(\mathsf{S}_{n+1}) = \mathrm{End}(\mathsf{S}_{n+1}^{\bar{0}}) \oplus \mathrm{End}(\mathsf{S}_{n+1}^{\bar{1}})$$

identifies $\mathbb{C}l(n)$ as a direct sum of two matrix algebras. Hence there are two classes of irreducible ungraded $\mathbb{C}l(n)$-modules (given by $\mathsf{S}_{n+1}^{\bar{0}}$ and $\mathsf{S}_{n+1}^{\bar{1}}$). These are distinguished by the action of the chirality element Γ_n (note that the map to $\mathbb{C}l^{\bar{0}}(n+1)$ takes Γ_n to Γ_{n+1}, up to sign).

It remains to classify irreducible $\mathbb{Z}_2$-graded $\mathbb{C}l(n)$-modules $E = E^{\bar{0}} \oplus E^{\bar{1}}$, for n odd. If $n = 1$, since $\dim \mathbb{C}l(1) = 2$ the Clifford algebra $\mathbb{C}l(1)$ itself is an example. Conversely, if E is an irreducible $\mathbb{C}l(1)$-module, the choice of any non-zero element $\phi \in E^{\bar{0}}$ defines an isomorphism $\mathbb{C}l(1) \to E,\ x \mapsto \varrho(x)\phi$. For general odd n, write $\mathbb{C}l(n) = \mathbb{C}l(n-1) \otimes \mathbb{C}l(1)$. If E is an irreducible $\mathbb{Z}_2$-*graded* $\mathbb{C}l(n)$-module, then $E_1 = \mathrm{Hom}_{\mathbb{C}l(n-1)}(\mathsf{S}_{n-1}, E)$ is a $\mathbb{Z}_2$-graded $\mathbb{C}l(1)$-module. This gives a decomposition

$$E \cong \mathsf{S}_{n-1} \otimes \mathrm{Hom}_{\mathbb{C}l(n-1)}(\mathsf{S}_{n-1}, E)$$

as $\mathbb{Z}_2$-graded $\mathbb{C}l(n-1) \otimes \mathbb{C}l(1) = \mathbb{C}l(n)$-modules (using graded tensor products). Since E is irreducible, the $\mathbb{Z}_2$-graded $\mathbb{C}l(1)$-module E_1 must be irreducible, hence it is isomorphic to $\mathbb{C}l(1)$. This proves that $E \cong \mathsf{S}_{n-1} \otimes \mathbb{C}l(1)$ as a $\mathbb{Z}_2$-graded $\mathbb{C}l(n-1) \otimes \mathbb{C}l(1) = \mathbb{C}l(n)$-module. □

REMARK 7.11 (Restrictions). Any $\mathbb{C}l(n)$-module can be regarded as a $\mathbb{C}l(n-1)$-module by restriction. By dimension count, one verifies:

(1) If n is even, then the ungraded module S_n restricts to a direct sum of the two non-isomorphic ungraded $\mathbb{C}l(n-1)$-modules (given by the even and odd part of S_n). The two $\mathbb{Z}_2$-graded modules $\mathsf{S}_n, \mathsf{S}_n^{\mathrm{op}}$ both become isomorphic to the unique $\mathbb{Z}_2$-graded module over $\mathbb{C}l(n-1)$.

(2) If n is odd, then the restrictions of the two irreducible ungraded $\mathbb{C}l(n)$-modules to $\mathbb{C}l(n-1)$ are both isomorphic to S_{n-1}, while the restriction of the irreducible $\mathbb{Z}_2$-graded $\mathbb{C}l(n)$-module is isomorphic to a direct sum $\mathsf{S}_{n-1} \oplus \mathsf{S}_{n-1}^{\mathrm{op}}$.

7.5. Spin representation. We saw that up to isomorphism, the algebra $\mathbb{C}l^{\bar{0}}(V)$ has two irreducible modules if $n = \dim V$ is even, and a unique such if n is odd. These restrict to representations of the group $\mathrm{Spin}(V) \subset \mathbb{C}l^{\bar{0}}(V)$, called the two *half-spin representations* if n is even, respectively the *spin representation* if n is odd. If $V = \mathbb{R}^n$, it is customary to denote the two half-spin representations (for n even) by $\Delta_n^{\pm}$, and the spin representation (for n odd) by Δ_n. Here Δ_n^+ (resp. Δ_n^-) is the half-spin representation where Γ_n acts as $+1$ (resp. as -1).

More concretely, taking $V = \mathbb{R}^{2m}$ with the spin representation defined by its standard complex structure $Je_{2j-1} = e_{2j}$, we may take $\Delta_{2m}^{\pm}$ to be the even and odd part of $\mathsf{S}_{2m} = \wedge\mathbb{C}^m$, and $\Delta_{2m-1} = \mathsf{S}_{2m-2} = \wedge\mathbb{C}^{m-1}$ (the spinor module over $\mathrm{Cl}(2m-2) \cong \mathrm{Cl}^{\bar{0}}(2m-1)$).

PROPOSITION 7.12. (i) *If n is even, the two half-spin representations $\Delta_n^{\pm}$ of* $\mathrm{Spin}(n)$ *are irreducible, and are non-isomorphic. Their restrictions to* $\mathrm{Spin}(n-1)$ *are both isomorphic to* Δ_{n-1}.

(ii) *If n is odd, the spin representation Δ_n is irreducible. Its restriction to* $\mathrm{Spin}(n-1)$ *is isomorphic to* $\Delta_{n-1} = \Delta_{n-1}^{+} \oplus \Delta_{n-1}^{-}$.

PROOF. This is immediate from the classification of irreducible $\mathrm{Cl}^{\bar{0}}(n)$-modules, since $\mathrm{Spin}(n)$ generate $\mathrm{Cl}^{\bar{0}}(n)$ as an algebra. (Note e.g. that $\mathrm{Spin}(n)$ contains the basis e_I of $\mathrm{Cl}^{\bar{0}}(n)$, where I ranges over subsets of $\{1,\ldots,n\}$ with an even number of elements.) ☐

We recall some terminology from the representation theory of compact Lie groups G (see e.g. [**15, 26**]). Let H be a Hermitian vector space carrying a unitary G-representation. The inner product on H will be denoted $\langle\cdot,\cdot\rangle$.

(i) H is of *real type* if it admits a G-equivariant conjugate linear endomorphism C with $C^2 = I$. In this case, H is the complexification of the real G-representation $\mathsf{H}_{\mathbb{R}}$ given as the fixed point set of C. Representations of real type carry a non-degenerate symmetric bilinear form $(\phi,\psi) = \langle C\phi,\psi\rangle$. Conversely, given a G-invariant non-degenerate skew-symmetric bilinear form, define a conjugate linear endomorphism T by $(\phi,\psi) = \langle T\phi,\psi\rangle$. The square $T^2 \in \mathrm{End}(\mathsf{H})$ is $\mathbb{C}$-linear, and is positive definite. Hence $|T| = (T^2)^{1/2}$ commutes with T, and $C = T|T|^{-1}$ defines a real structure. We will call the bilinear form *compatible* with the Hermitian structure if $C = T$.

(ii) H is of *quaternionic type* if it admits a G-equivariant conjugate linear endomorphism C with $C^2 = -I$. In this case, C gives H the structure of a quaternionic G-representation, where scalar multiplication by the quaternions i,j,k is given by $i = \sqrt{-1},\ j = C,\ k = ij$. From C one obtains a non-degenerate skew-symmetric bilinear form $(\phi,\psi) = \langle C\phi,\psi\rangle$. Conversely, given a G-invariant non-degenerate symmetric bilinear form, define a conjugate linear endomorphism T by $(\phi,\psi) = \langle T\phi,\psi\rangle$. Again, $|T| = (-T^2)^{1/2}$ commutes with T, and $C = T|T|^{-1}$ defines a quaternionic structure. We will call the bilinear form *compatible* with the Hermitian structure if $C = T$.

(iii) For G-representations H of real or quaternionic type, the structure map C gives an isomorphism with the dual G-representation H^*. That is, such representations are self-dual. We will call a unitary G-representation of *complex type* if it is not self-dual.

For a real or quaternionic representation, the corresponding bilinear form defines an element of $\mathrm{Hom}_G(\mathsf{H},\mathsf{H}^*)$. If H is irreducible, then this space is 1-dimensional. Hence for irreducible representations the real and quaternionic case are exclusive. This proves part of the following result (see [**15**, Chapter II.6]).

THEOREM 7.13. *If* H *is an* irreducible *unitary G-representation, then it is either of real, complex or quaternionic type.*

We now specialize to the spin representations. The canonical bilinear form on spinor modules S can be viewed as scalar-valued, after choice of a trivialization of the canonical line K_{S}.

PROPOSITION 7.14. *Let V be a Euclidean vector space of even dimension $n = 2m$, and S a unitary spinor module over $\mathbb{C}l(V)$. Then the Hermitian metric on S and the canonical bilinear form are compatible.*

PROOF. We work in the model $V^{\mathbb{C}} = F^* \oplus F$, $\mathsf{S} = \wedge F^*$. Let $f^1, \ldots, f^m$ be a basis of F^*, orthonormal relative to the Hermitian inner product. Then the f^I for subsets $I \subset \{1, \ldots, m\}$ define an orthonormal basis of $\wedge F^*$. For any subset I let I^c be the complementary subset. Use $f^1 \wedge \cdots \wedge f^m$ to trivialize $\det(F^*)$, and define signs $\epsilon_I = \pm 1$ by

$$(f^I)^\top \wedge f^{I^c} = \epsilon_I \, f^1 \wedge \cdots \wedge f^m.$$

Thus $(f^I, f^{I^c})_{\mathsf{S}} = \epsilon_I$. Notice that $\epsilon_I = \epsilon_{I^c}$ in the symmetric case and $\epsilon_I = -\epsilon_{I^c}$ in the skew-symmetric case. Define C by

$$(f^I, f^J)_{\mathsf{S}} = \langle C f^I, f^J \rangle$$

Then $C f^I = \epsilon_I f^{I_c}$. We read off that $C^2 = I$ in the symmetric case and $C^2 = -I$ in the skew-symmetric case. ☐

THEOREM 7.15. *The types of the spin representations of* $\mathrm{Spin}(n)$ *are as follows.*

$$\begin{aligned}
n &= 0 \mod 8: &&\Delta_n^\pm \ \textit{real type,}\\
n &= 1, 7 \mod 8: &&\Delta_n \ \textit{real type,}\\
n &= 2, 6 \mod 8: &&\Delta_n^\pm \ \textit{complex type,}\\
n &= 3, 5 \mod 8: &&\Delta_n \ \textit{quaternionic type,}\\
n &= 4 \mod 8: &&\Delta_n^\pm \ \textit{quaternionic type.}
\end{aligned}$$

If $n = 2, 6 \mod 8$, *one has* $\Delta_n^- \cong (\Delta_n^+)^*$.

PROOF. We consider various subcases.

Case 1a: $n = 2m$ with m even. The canonical bilinear form $(\cdot, \cdot)_S$ is still non-degenerate on the even and odd part of the spinor module; hence it defines a real structure if $m = 0 \mod 4$ and a quaternionic structure for $m = 2 \mod 4$.

Case 1b: $n = 2m$ with m odd. Then $(\cdot, \cdot)_S$ defines a non-degenerate pairing between $\mathsf{S}_n^{\bar{0}} = \Delta_n^+$ and $\mathsf{S}_n^{\bar{1}} = \Delta_n^-$. Hence $\Delta_n^\pm \not\cong \Delta_n^\mp = (\Delta_n^\pm)^*$, so that $\Delta_n^\pm$ are of complex type.

Case 2a: $n = 2m - 1$ is odd, with m even. Recall that $\Delta_n = \mathsf{S}_{n+1}^{\bar{0}}$. The restriction of $(\cdot, \cdot)_{\mathsf{S}}$ gives the desired non-degenerate symmetric (if $m = 0 \mod 4$) or skew-symmetric (if $m = 2 \mod 4$) bilinear form.

Case 2b: $n = 2m - 1$ with m is odd. Here the restriction of $(\cdot,\cdot)_{\mathsf{S}}$ to $\mathsf{S}^{\bar{0}}_{n+1}$ is zero. We instead use the form

$$(\phi, \psi)' := (\phi,\ \varrho(e_{n+1})\psi)_{\mathsf{S}} \equiv (\varrho(e_{n+1})\phi, \psi)_{\mathsf{S}},$$

for $\phi, \psi \in \Delta_{2m-1} = \Delta_{2m}$. This is no longer $\mathrm{Spin}(n+1)$-invariant, but is still $\mathrm{Spin}(n)$-invariant. Since

$$\begin{aligned}(\psi, \phi)' &= (\varrho(e_{n+1})\psi,\ \phi)_{\mathsf{S}}\\ &= (-1)^{m(m-1)/2}(\phi,\ \ \varrho(e_{n+1})\psi)_{\mathsf{S}}\\ &= (-1)^{m(m-1)/2}(\phi, \psi)'.\end{aligned}$$

we find that the bilinear form is symmetric for $m = 1 \mod 4$ and skew-symmetric for $m = 3 \mod 4$.

□

7.6. Applications to compact Lie groups. Theorem 7.15 has severalimportant Lie-theoretic implications. **Spin(3).** The spin representation Δ_3 has dimension 2, and after choice of basis defines a homomorphism $\mathrm{Spin}(3) \to \mathrm{U}(2)$. Since $\mathrm{Spin}(3)$ is semi-simple, the image lies in $\mathrm{SU}(2)$, and by dimension count the resulting map $\mathrm{Spin}(3) \to \mathrm{SU}(2)$ must an isomorphism. Since Δ_3 is of quaternionic type, one similarly has a homomorphism $\mathrm{Spin}(3) \to \mathrm{Aut}(\mathbb{H}) = \mathrm{Sp}(1)$, which is an isomorphism by dimension count. That is, we recover $\mathrm{Spin}(3) \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1)$.

Spin(4). The two half-spin representations $\Delta_4^{\pm}$ are both 2-dimensional. Arguing as for $n = 3$, we see that the representation on $\Delta_4^+ \oplus \Delta_4^-$ defines an isomorphism $\mathrm{Spin}(4) \to \mathrm{SU}(2) \times \mathrm{SU}(2)$.

Spin(5). Δ_5 is a 4-dimensional representation of quaternionic type. After choice of an orthonormal quaternionic basis, this gives a homomorphism to $\mathrm{Sp}(2) = \mathrm{Aut}(\mathbb{H}^2)$, which by dimension count must be an isomorphism. This realizes the isomorphism

$$\mathrm{Spin}(5) \cong \mathrm{Sp}(2).$$

In particular, we see that $\mathrm{Spin}(5)$ acts transitively on the unit sphere $S^7 \subset \Delta_5 \cong \mathbb{C}^4$, since $\mathrm{Sp}(2)$ acts transitively on the quaternions of unit norm. The stabilizer of the base point $(0, 1) \in \mathbb{H}^2$ is $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ (embedded in $\mathrm{Sp}(2)$ as the upper left block). We hence see that

$$S^7 = \mathrm{Spin}(5)/\,\mathrm{SU}(2).$$

This checks with dimensions, since $\dim \mathrm{Spin}(5) = 10$, while $\mathrm{SU}(2)$ has dimension 3.

Spin(6). The half-spin representations $\Delta_6^\pm$ are 4-dimensional, and define homomorphisms $\mathrm{Spin}(6) \to \mathrm{U}(4)$. Since $\mathrm{Spin}(6)$ is semi-simple, this homomorphism must take values in $\mathrm{SU}(4)$, realizing the isomorphism

$$\mathrm{Spin}(6) \cong \mathrm{SU}(4).$$

In particular $\mathrm{Spin}(6)$ acts transitively on $S^7 \subset \Delta_6^\pm \cong \Delta_5$, extending the action of $\mathrm{Spin}(5)$, with stabilizers $\mathrm{SU}(3)$.

Spin(7). The 8-dimensional spin representation Δ_7 is of real type, hence it can be regarded as the complexification of an 8-dimensional real representation $\Delta_7^{\mathbb{R}} \cong \mathbb{R}^8$. Restricting to $\mathrm{Spin}(6)$, we have $\Delta_7 = \Delta_6^+ \oplus \Delta_6^-$. Under the symmetric bilinear form on Δ_7, both $\Delta_6^\pm$ are Lagrangian. [1] This implies $\Delta_7^{\mathbb{R}} \cong \Delta_6^\pm$ as real $\mathrm{Spin}(6) \subset \mathrm{Spin}(7)$-representations. Since $\mathrm{Spin}(6)$ acts transitively on the unit sphere $S^7 \subset \Delta_6^\pm$, this shows that $\mathrm{Spin}(7)$ acts transitively on the unit sphere $S^7 \subset \Delta_7^{\mathbb{R}}$. Let H be the isotropy at some given base point on S^7. It is a compact Lie group of dimension

$$\dim H = \dim \mathrm{Spin}(7) - \dim S^7 = 21 - 7 = 14.$$

More information is obtained using some homotopy theory. For a compact, simple simply connected simple Lie group G one knows that $\pi_1(G) = \pi_2(G) = 0$, while $\pi_3(G) = \mathbb{Z}$. By the long exact sequence of homotopy groups of a fibration,

$$\cdots \to \pi_{k+1}(S^7) \to \pi_k(H) \to \pi_k(\mathrm{Spin}(7)) \to \pi_k(S^7) \to \cdots,$$

and using that $\pi_k(S^7) = 0$ for $1 < k < 7$ (Hurewicz' theorem), we find that $\pi_1(H) = \pi_2(H) = 0$ and $\pi_3(H) = \mathbb{Z}$. It follows that H is simply connected and simple (otherwise $\pi_3(H)$ would have more summands). But in dimension 14 there is a unique such group: the exceptional Lie group G_2. This proves the following remarkable result.

THEOREM 7.16. *There is a transitive action of* $\mathrm{Spin}(7)$ *on* S^7*. The stabilizer subgroups for this action are isomorphic to the exceptional Lie group* G_2*. That is,*

$$S^7 = \mathrm{Spin}(7)/G_2.$$

We remark that one can also directly identify the root system for H, avoiding the use of algebraic topology or appealing to the classification of Lie groups. This is carried out in Adams' book [**1**, Chapter 5].

Spin(8). The triality principle from Section 6 specializes to give a degree 3 automorphism j of the group $\mathrm{Spin}(8, \mathbb{C})$, along with a degree 3 automorphism J of $\mathbb{C}^8 \oplus \Delta_8^+ \oplus \Delta_8^-$ interchanging the three summands, such that the

[1] In more detail, recall that the bilinear form on $\Delta_7 \cong \mathsf{S}_8^{\bar{0}}$ is $\mathbb{C}l(7) \cong \mathbb{C}l^0(8)$-invariant. Restricting to $\mathbb{C}l(6) \subset \mathbb{C}l(7)$, we obtain a $\mathbb{C}l(6)$-invariant bilinear form on $\Delta_6^+ \oplus \Delta_6^- \cong \mathsf{S}_6$, which must agree with the canonical bilinear form up to scalar multiple. But the latter vanishes on the even and odd part of S_6.

induced maps $J\colon \Delta_8^- \to \mathbb{C}^8$ etc. are equivariant relative to the automorphism j. Since $\Delta_8^\pm$ are of real type, one may hope for J to preserve the real subspace $\mathbb{R}^8 \oplus \Delta_8^{+,\mathbb{R}} \oplus \Delta_8^{-,\mathbb{R}}$, and for j to preserve Spin(8). This is accomplished by taking the vectors n, q in the construction of J, j (see Section 6) to lie in $\mathbb{R}^8$ and $\Delta_8^{+,\mathbb{R}}$, respectively. The trilinear form on $\mathbb{C}^8 \oplus \Delta_8^+ \oplus \Delta_8^-$ (cf. (45)) restricts to the real part, and can be used [**17, 22**] to define on $\mathbb{R}^8$ an octonion multiplication, $\mathbb{R}^8 \cong \mathbb{O}$. The exceptional group G_2 is now realized as the automorphism group $\mathrm{Aut}(\mathbb{O})$ of the octonions. A beautiful survey of this theory is given in Baez' article [**9**].

The other exceptional groups E_6, E_7, E_8, F_4 are related to spin groups as well. For example, F_4 contains a copy of Spin(9), and the action of Spin(9) on $\mathfrak{f}_4/\mathfrak{o}(9)$ is isomorphic to the (real) spin representation $\Delta_9^{\mathbb{R}}$. In a similar fashion, E_8 contains a copy of $\mathrm{Spin}(16)/\mathbb{Z}_2$ (where $\mathbb{Z}_2$ is generated by the chirality element Γ_{16}), and the action of $\mathrm{Spin}(16)/\mathbb{Z}_2$ on the quotient $\mathfrak{e}_8/\mathfrak{o}(16)$ is isomorphic to the (real) spin representation of Spin(16) on $\Delta_{16}^{+,\mathbb{R}}$. (This checks with dimensions: E_8 is 248-dimensional, Spin(16) is 120-dimensional, and $\Delta_{16}^{+,\mathbb{R}}$ is $2^7 = 128$-dimensional.) Proofs, and a wealth of related results, can be found in Adams' book [**1**].

CHAPTER 4

Covariant and contravariant spinors

Suppose V is a quadratic vector space. In §2, Section 2.11 we defined a map $\lambda\colon \mathfrak{o}(V) \to \wedge^2(V)$, which is a Lie algebra homomorphism relative to the Poisson bracket on $\wedge(V)$, and a map $\gamma = q \circ \lambda\colon \mathfrak{o}(V) \to \mathrm{Cl}(V)$, which is a Lie algebra homomorphism relative to the Clifford commutator. One of the problems addressed in this chapter is to give explicit formulas for the Clifford exponential $\exp(\gamma(A)) \in \mathrm{Cl}(V)$. We will compute its image under the symbol map, and express its relation to the exterior algebra exponential $\exp(\lambda(A))$. These questions will be studied using the spin representation for $W = V^* \oplus V$, with bilinear form given by the pairing. The shift in perspective is reflected by a mild change of notation: Rather than starting with a split bilinear form on V and using a decomposition of $V = F^* \oplus F$ into Lagrangian subspaces $F, F' \cong F^*$, we now consider V itself as a Lagrangian subspace of $W = V^* \oplus V$.

1. Pull-backs and push-forwards of spinors

Let V be any vector space, and let $W = V^* \oplus V$ carry as usual the bilinear form

$$B_W((\mu_1, v_1), (\mu_2, v_2)) = \tfrac{1}{2}(\langle \mu_1, v_2\rangle + \langle \mu_2, v_1\rangle). \tag{48}$$

We will occasionally use a basis $e_1, \dots, e_m$ of V, with dual basis $f^1, \dots, f^m$ of V^*. Thus $B_W(e_i, f^j) = \frac{1}{2}\delta_i^j$, and the Clifford relations in $\mathrm{Cl}(W)$ read, in terms of super commutators,

$$[f^i, f^j] = 0, \quad [e_i, f^j] = \delta_i^j, \quad [e_i, e_j] = 0.$$

We define the standard or *contravariant* spinor module to be $\wedge(V^*)$, with generators $\mu \in V^*$ acting by exterior multiplication and $v \in V$ acting by contraction. We will also consider the dual or *covariant* spinor module $\wedge(V)$, with generators $v \in V$ acting by exterior multiplication and $\mu \in V^*$ acting by contraction. Recall (cf. §3.4) that there is a canonical isomorphism of Clifford modules,

$$\wedge(V^*) \cong \wedge(V) \otimes \det(V^*),$$

defined by contraction. The choice of a generator $\Gamma_\wedge \in \det(V^*)$ gives an isomorphism, called the 'star operator' for the volume form $\Gamma_\wedge$

$$*_{\Gamma_\wedge}\colon \wedge(V) \to \wedge(V^*),\ \chi \mapsto \iota(\chi)\Gamma_\wedge.$$

Thus

$$*_{\Gamma_\wedge} \circ \epsilon(v) = \iota(v) \circ *_{\Gamma_\wedge},$$
$$*_{\Gamma_\wedge} \circ \iota(\mu) = \epsilon(\mu) \circ *_{\Gamma_\wedge},$$

In **??**, we saw that the most general contravariant pure spinor is of the form

$$\phi = \exp(-\omega)\kappa, \tag{49}$$

where $\omega \in \wedge^2 V^*$ is a 2-form, and $\kappa \in \det(\mathrm{ann}(N))^\times$ a volume form on V/N, for some subspace $N \subset V$. Given κ, the form ϕ only depends on the restriction $\omega_N \in \wedge^2(N^*)$ of ω to N.

By reversing the roles of V, V^*, the most general covariant spinor is of the form

$$\chi = \exp(-\pi)\nu \tag{50}$$

where $\pi \in \wedge^2(V)$ and $\nu \in \det(S)^\times$, for some subspace $S \subset V$. Given ν, the spinor χ only depends on the image $\pi_S \in \wedge^2(V/S)$ of π. The corresponding Lagrangian subspace is

$$F(e^{-\pi}\nu) = \{(\mu, v) \in V^* \oplus V |\ \mu \in \mathrm{ann}(S),\ \pi(\mu, \cdot) - v \in S\}. \tag{51}$$

Note that $S = F(e^{-\pi}\nu) \cap V$, while $\mathrm{ann}(S)$ is characterized as projection of $F(e^{-\pi}\nu)$ to V^*.

For a linear map $\Phi\colon V_1 \to V_2$ we denote by $\Phi_* = \wedge(\Phi)\colon \wedge V_1 \to \wedge V_2$ the 'push-forward' map, and by $\Phi^* = \wedge(\Phi^*)\colon \wedge V_2^* \to \wedge V_1^*$ the 'pull-back' map.

PROPOSITION 1.1 (Push-forwards). *Suppose $\Phi\colon V_1 \to V_2$ is a linear map, and $\chi \in \wedge V_1$ is a pure spinor. Then the following are equivalent:*

1. $\Phi_*\chi \neq 0$,
2. $\ker(\Phi) \cap \{v_1 |\ v_1 \wedge \chi = 0\} = \{0\}$,
3. $\Phi_*\chi$ *is a pure spinor.*

In this case, the Lagrangian subspace defined by the pure spinor $\Phi_*\chi$ is

$$F(\Phi_*\chi) = \{(\mu_2, \Phi_* v_1) |\ (\Phi^*\mu_2, v_1) \in F(\chi)\}.$$

PROOF. Write $\chi = e^{-\pi}\nu$ as in (50). Since ν is a generator of $\det(S)$, the subspace $S \subset V_1$ may characterized in terms as the set of all $v_1 \in V_1$ with $v_1 \wedge \nu = 0$, that is $v_1 \wedge \chi = 0$. Thus $\Phi_*\chi \neq 0 \Leftrightarrow \Phi_*\nu \neq 0 \Leftrightarrow \ker(\Phi) \cap S = \{0\}$. Furthermore, in this case $\Phi_*\nu$ is a generator of $\det(\Phi(S))$, and hence $\Phi_*\chi = e^{-\Phi_*\pi}\Phi_*\nu$ is a pure spinor.

We have $(\mu_2, v_2) \in F(\Phi_*\chi)$ if and only if $\mu_2 \in \mathrm{ann}(\Phi(S))$ and $\Phi_*\pi(\mu_2, \cdot) - v_2 \in \Phi(S)$. The first condition means $\Phi^*\mu_2 \in \mathrm{ann}(S)$. Choose $w_1 \in V_1$ with $(\Phi^*\mu_2, w_1) \in F(\chi)$. (Note that w_1 is unique modulo $F(\chi) \cap V_1 = S$.) Then $\pi(\Phi^*\mu_2, \cdot) - w_1 \in S$, i.e. $\Phi_*\pi(\mu_2, \cdot) - \Phi(w_1) \in \Phi(S)$. The second condition now shows that $v_2 - \Phi(w_1) = \Phi(u_1)$ for a unique element $u_1 \in S$. Putting $v_1 = w_1 + u_1$, we obtain $v_2 = \Phi(v_1)$, giving the desired description of $F(\Phi_*\chi)$. ∎

By a similar argument, one shows:

PROPOSITION 1.2 (Pull-backs). *Suppose* $\Phi\colon V_1 \to V_2$ *is a linear map, and* $\phi \in \wedge V_2^*$ *is a pure spinor. Then the following are equivalent:*

(1) $\Phi^*\phi \neq 0$,
(2) $\ker(\Phi^*) \cap \{\mu_2 \in V_2^* |\ \mu_2 \wedge \phi = 0\} = \{0\}$,
(3) $\Phi^*\phi$ *is a pure spinor.*

In this case, the Lagrangian subspace defined by $\Phi^*\phi$ *is*

$$F(\Phi^*\phi) = \{(\Phi^*\mu_2, v_1)|\ (\mu_2, \Phi_* v_1) \in F(\phi)\}.$$

The two Propositions suggest notions of push-forwards and pull-backs of Lagrangian subspaces. Write $(\mu_1, v_1) \sim_\Phi (\mu_2, v_2)$ if $v_2 = \Phi(v_1)$ and $\mu_1 = \Phi^*(\mu_2)$. If $E \subset V_1^* \oplus V_1$ is Lagrangian we define the *forward image of* E

$$\Phi_! E = \{(\mu_2, v_2) \in V_2^* \oplus V_2|\ \exists (\mu_1, v_1) \in E\colon (\mu_1, v_1) \sim_\Phi (\mu_2, v_2)\}.$$

If $F \subset V_2^* \oplus V_2$ is Lagrangian we define the *backward image of* F

$$\Phi^! F = \{(\mu_1, v_1) \in V_1^* \oplus V_1|\ \exists (\mu_2, v_2) \in F\colon (\mu_1, v_1) \sim_\Phi (\mu_2, v_2)\}.$$

The two Propositions above show that $\Phi_! E$ is Lagrangian, provided $\ker(\Phi) \cap (E \cap V_1) = \{0\}$, and that $\Phi^! F$ is Lagrangian provided $\ker(\Phi^*) \cap (F \cap V_1^*) = \{0\}$. In fact, $\Phi_! E$ and $\Phi^! F$ are Lagrangian even without these transversality assumptions. However, assuming that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the forward image $\Phi_! E$ (resp. backward image $\Phi^! F$) depends continuously on E (resp. on F) only on the open subset of $\mathrm{Lag}(V^* \oplus V)$ where the transversality condition is satisfied.

PROPOSITION 1.3. *Let* $\Phi\colon V_1 \to V_2$ *be a linear map. Suppose* E_1 *is a Lagrangian subspace of* $V_1^* \oplus V_1$, *and* F_2 *is a Lagrangian subspace of* $V_2^* \oplus V_2$. *Let* $E_2 = \Phi_! E_1$ *be the forward image of* E_1 *and* $F_1 = \Phi^! F_2$ *the backward image of* F_2. *Then*

$$E_1 \cap F_1 = \{0\} \quad \Leftrightarrow \quad E_2 \cap F_2 = \{0\}.$$

Furthermore, in this case the transversality conditions are automatic, that is $\ker(\Phi) \cap (E_1 \cap V_1) = \{0\}$ *and* $\ker(\Phi^*) \cap (F_2 \cap V_2^*) = \{0\}$.

PROOF. Consider the covariant spinor module over $\mathrm{Cl}(V_1^* \oplus V_1)$ and the contravariant one over $\mathrm{Cl}(V_2^* \oplus V_2)$. Let $\chi \in \wedge V_1$ be a pure spinor defining E_1, and $\phi \in \wedge V_2^*$ a pure spinor defining F_2. Then $\langle \phi, \Phi_* \chi \rangle = \langle \Phi^* \phi, \chi \rangle$, hence one pairing vanishes if and only if the other pairing vanishes. But by Theorem 4.7 (or rather its proof), the non-vanishing of the pairing is equivalent to transversality of the corresponding Lagrangian subspaces. Furthermore, the non-vanishing implies that $\Phi_* \chi$, $\Phi^* \phi$ are both non-zero, which by Propositions 1.1 and 1.2 gives the transversality conditions. □

2. Factorizations

2.1. The Lie algebra $\mathfrak{o}(V^* \oplus V)$. Let $W = V^* \oplus V$, and recall the isomorphism (cf. §2, Section 2.11)

$$\lambda\colon \mathfrak{o}(W) \to \wedge^2(W),$$

given in terms of the Poisson bracket on $\wedge(W)$ by $S(w) = \{\lambda(S), w\}$. We are interested in the action of $\gamma(S) = q(\lambda(S)) \in \mathrm{Cl}(W)$ in the contravariant spinor module $\wedge(V^*)$. To compute this action, decompose

$$\wedge^2(W) = \wedge^2(V) \oplus \wedge^2(V^*) \oplus (V^* \wedge V).$$

In $\mathfrak{o}(W)$, the three summands correspond to:

1. The commutative Lie algebra $\wedge^2(V)$ of skew-adjoint maps $E_1\colon V^* \to V$, acting as $(\mu, v) \mapsto (0, E_1(\mu))$. Equivalently, this is the subalgebra of $\mathfrak{o}(W)$ fixing V pointwise. We have
$$\lambda(E_1) = \tfrac{1}{2} \sum_i E_1(f^i) \wedge e_i. \tag{52}$$
The element $\gamma(E_1) \in \wedge^2(V)$ is given by the same formula (viewing $\wedge(V)$ as a subalgebra of $\mathrm{Cl}(W)$), and its action in the spinor module is by contraction with $\lambda(E_1)$.
2. The commutative Lie algebra $\wedge^2(V^*)$ of skew-adjoint maps $E_2\colon V \to V^*$, acting as $(\mu, v) \mapsto (E_2(v), 0)$. Equivalently, this is the subalgebra of $\mathfrak{o}(W)$ fixing V^* pointwise. We have
$$\lambda(E_2) = \tfrac{1}{2} \sum_i E_2(e_i) \wedge f^i.$$
The element $\gamma(E_2) \in \wedge^2(V^*)$ is given by the same formula (viewing $\wedge(V^*)$ as a subalgebra of $\mathrm{Cl}(W)$), and its action in the spinor module is by exterior multiplication by $\lambda(E_2)$.
3. The Lie algebra $\mathfrak{gl}(V)$, where $A\colon V \to V$ acts as $(\mu, v) \mapsto (-A^*\mu, Av)$. Equivalently, this is the subalgebra of $\mathfrak{o}(W)$ preserving the direct sum decomposition $V^* \oplus V$. We have
$$\lambda(A) = -\sum_i f^i \wedge A(e_i).$$
This quantizes to
$$\begin{aligned}\gamma(A) = q(\lambda(A)) &= -\tfrac{1}{2} \sum_i (f^i A(e_i) - A(e_i) f^i)\\ &= -\sum_i f^i A(e_i) + \tfrac{1}{2}\operatorname{tr}(A)\end{aligned}$$
where we used $\sum_i \langle f^i,\ A(e_i)\rangle = \operatorname{tr}(A)$. Letting L_A denote the canonical action of $\mathfrak{gl}(V)$ on $\wedge(V^*)$, given as the derivation extensions of the action as $-A^*$ on V^*, we find that $\gamma(A)$ acts as
$$\varrho(\gamma(A)) = L_A + \tfrac{1}{2}\operatorname{tr}(A).$$

That is, the action of $\mathfrak{gl}(V)$ on the spinor module $\wedge(V^*)$ differs from the 'standard' action by the 1-dimensional character $A \mapsto \frac{1}{2}\operatorname{tr}(A)$.

Writing elements of $W = V^* \oplus V$ as column vectors, we see that

$$\mathfrak{o}(W) \cong \wedge^2(V^*) \oplus \mathfrak{gl}(V) \oplus \wedge^2(V)$$

consists of block matrices of the form

$$S = \begin{pmatrix} -A^* & E_2 \\ E_1 & A \end{pmatrix},$$

and our discussion shows

$$\varrho(\gamma(S)) = \iota(\lambda(E_1)) + \epsilon(\lambda(E_2)) + L_A + \tfrac{1}{2}\operatorname{tr}(A).$$

2.2. The group $\mathrm{SO}(V^* \oplus V)$. Corresponding to the three Lie subalgebras of $\mathfrak{o}(W)$ there are three subgroups of $\mathrm{SO}(W)$:

1. $\wedge^2(V^*)$, given as matrices in block form
$$\begin{pmatrix} I & E_2 \\ 0 & I \end{pmatrix}$$
where $E_2 \colon V \to V^*$ is a skew-adjoint linear map.
2. $\wedge^2(V)$, given as matrices in block form
$$\begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix}$$
where $E_1 \colon V^* \to V$ is a skew-adjoint linear map.
3. $\mathrm{GL}(V)$, embedded as matrices in block form
$$\begin{pmatrix} (Q^{-1})^* & 0 \\ 0 & Q \end{pmatrix}$$
where $Q \colon V \to V$ is invertible.

If $\mathbb{K} = \mathbb{R}, \mathbb{C}$, these are Lie subgroups of $\mathrm{SO}(W)$, with respective Lie subalgebras $\wedge^2(V^*)$, $\mathfrak{gl}(V)$, $\wedge^2(V)$ of $\mathfrak{o}(W)$. Consider now arbitrary orthogonal transformations. An endomorphism $g \in \mathrm{End}(W)$, written in block form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{53}$$

is orthogonal if and only if

$$a^*c + c^*a = 0, \quad b^*d + d^*b = 0, \quad a^*d + c^*b = I.$$

Proposition 2.1 (Factorization formulas).

(1) *The map $\wedge^2(V) \times \mathrm{GL}(V) \times \wedge^2(V^*) \to \mathrm{SO}(W)$ taking (g_1, g_2, g_3) to the product $g = g_1 g_2 g_3$ is injective. Its image is the set of all orthogonal transformations* (53) *for which the block $a \in \mathrm{End}(V^*)$ is invertible.*

(2) *The map $\wedge^2(V^*) \times \mathrm{GL}(V) \times \wedge^2(V) \to \mathrm{SO}(W)$ taking (g_1, g_2, g_3) to $g_1 g_2 g_3$ is injective. Its image is the set of all orthogonal transformations* (53) *for which the block $d \in \mathrm{End}(V)$ is invertible.*

In particular, the orthogonal transformations (53) for which the block a or the block d are invertible, are contained in $\mathrm{SO}(V^* \oplus V)$.

Proof. For (1) we want to write (53) as a product

$$\begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix}\begin{pmatrix} (Q^{-1})^* & 0 \\ 0 & Q \end{pmatrix}\begin{pmatrix} I & E_2 \\ 0 & I \end{pmatrix} = \begin{pmatrix} (Q^{-1})^* & (Q^{-1})^* E_2 \\ E_1(Q^{-1})^* & Q + E_1(Q^{-1})^* E_2 \end{pmatrix}.$$

If a is invertible, we can solve for E_1, E_2, Q in terms of the blocks a, b, c:

$$Q = (a^{-1})^*, \; E_1 = ca^{-1}, \; E_2 = a^{-1}b.$$

(Note that $d = (a^{-1})^*(I - c^*b)$ if a is invertible.) The proof of (2) is similar. □

2.3. The group $\mathrm{Spin}(V^* \oplus V)$**.** The factorizations of $\mathrm{SO}(W)$ give rise to factorizations of the special Clifford group $S\Gamma(W)$ and of the spin group $\mathrm{Spin}(W)$.

(1) The inclusion of $\wedge^2(V) \hookrightarrow \mathrm{SO}(W)$ lifts to an inclusion as a subgroup of $\mathrm{Spin}(W) \subset S\Gamma(W)$, by the map

$$\begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix} \mapsto \tilde{E}_1 := \exp(\lambda(E_1))$$

where the right hand side is an element of $\wedge(V^*) \subset \mathrm{Cl}(W)$. Indeed, for all $(\mu, v) \in W$ one has

$$\tilde{E}_1(\mu, v)\tilde{E}_1^{-1} = (\mu, v - \iota_\mu \lambda(E_1))$$

showing that $\tilde{E}_1$ lies in $S\Gamma(W)$ and that it lifts the orthogonal transformation defined by E_1. Since $\mathsf{N}(\tilde{E}_1) = 1$ it is an element of the spin group. The action of this factor in the spinor module is

$$\varrho(\tilde{E}_1)\phi = \iota(e^{\lambda(E_1)})\phi.$$

(2) Similarly, the group homomorphism $\wedge^2(V^*) \hookrightarrow \mathrm{SO}(W)$ lifts to an inclusion into $\mathrm{Spin}(W) \subset S\Gamma(W)$ by the map

$$\begin{pmatrix} I & E_2 \\ 0 & I \end{pmatrix} \mapsto \tilde{E}_2 := \exp(\lambda(E_2))$$

where the right hand side is viewed as an element of $\wedge(V^*) \subset \mathrm{Cl}(W)$. The action of this factor in the spinor module is

$$\varrho(\tilde{E}_2)\phi = e^{\lambda(E_2)} \wedge \phi.$$

(3) The inclusion $\mathrm{GL}(V) \subset \mathrm{SO}(W)$ does not have a natural lift. Let $\mathrm{GL}_\Gamma(V) \subset S\Gamma(W)$ denote the pre-image of $\mathrm{GL}(V)$, so that there is an exact sequence

$$1 \to \mathbb{K}^\times \to \mathrm{GL}_\Gamma(V) \to \mathrm{GL}(V) \to 1.$$

If $\tilde{Q} \in \mathrm{GL}_\Gamma(V)$ is a lift of $Q \in \mathrm{GL}(V)$, then its action in the spinor module reads,

$$\varrho(\tilde{Q})\phi = \chi(\tilde{Q})\; Q.\phi.$$

Here $Q.\phi$ denote the 'standard' action of $\mathrm{GL}(V)$ on $\wedge V^*$, given by the extension of $Q \mapsto (Q^{-1})^* \in \mathrm{End}(V^*)$ to an algebra homomorphism of $\wedge(V^*)$, while $\chi\colon \mathrm{GL}_\Gamma(V) \to \mathbb{K}^\times$ is the restriction of the character $\chi\colon S\Gamma(W)_V \to \mathbb{K}^\times$ from Section 5. Recall its property $\chi(\tilde{Q})^2 = \mathsf{N}(\tilde{Q})\det(Q)$. If $\tilde{Q}$ can be normalized to lie in $\mathrm{Spin}(W) \subset \Gamma(W)_V$, we have

$$\chi(\tilde{Q}) = \det{}^{1/2}(Q)$$

where the sign of the square root depends on the choice of lift.

Example 2.2. Suppose V is 1-dimensional, with generator e. Let $f \in V^*$ be the dual generator so that $B_W(e, f) = \frac{1}{2}$. The spinor module $\mathsf{S} = \wedge V^*$ has basis $\{1, f\}$. Given $r \in \mathbb{K}^\times \cong \mathrm{GL}(V)$, the possible lifts of

$$\begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix} \in \mathrm{SO}(W)$$

are given by,

$$\tilde{Q} = t\big(1 - (1 - r^{-1}) f e\big) \in S\Gamma(W)$$

where $t \in \mathbb{K}^\times$. One checks $\mathsf{N}(\tilde{Q}) = t^2 r^{-1}$. If r admits a square root (e.g. if $\mathbb{K} = \mathbb{C}$), one obtains two lifts $\tilde{Q} \in \mathrm{Spin}(W)$ (one for each choice of square root $t = r^{1/2}$). The action of $\tilde{Q}$ in the spin representation is given by

$$\varrho(\tilde{Q})1 = t, \quad \varrho(\tilde{Q})f = t r^{-1} f$$

which is consistent with the formula given above.

Remark 2.3. One can also consider the spin representation of $S\Gamma(W)$ on the *dual* spinor module $\mathsf{S}^V = \wedge(V)$. Here the the formulas for the action of the three factors read

$$\varrho(\tilde{E}_1)\psi = e^{\lambda(E_1)} \wedge \psi, \quad \varrho(\tilde{E}_2)\psi = \iota(e^{\lambda(E_2)})\,\psi, \quad \varrho(\tilde{Q})\psi = \frac{\chi(\tilde{Q})}{\det Q}\, Q_*\,\psi.$$

(Due to the factor $\det(Q)^{-1}$, the action of $\tilde{Q}$ on the pure spinor line $l_V = \det(V)$ is multiplication by $\chi(\tilde{Q})$.)

3. The quantization map revisited

Up to this point, we discussed spinors only for quadratic vector spaces whose bilinear form B is *split*. By the following construction, the spinor module may be used for Clifford algebras of arbitrary symmetric bilinear forms, even degenerate ones. In particular, we can interpret the symbol map in terms of the spinor module.

3.1. The symbol map in terms of the spinor module. Suppose V is a vector space with a symmetric bilinear form, B. Then the map

$$j:\, V \mapsto W := V^* \oplus V,\ v \mapsto B^\flat(v) \oplus v$$

where W carries the bilinear form B_W given by (48), is a partial isometry:

$$B_W(j(v_1), j(v_2)) = \tfrac{1}{2}\langle B^\flat(v_1), v_2\rangle + \tfrac{1}{2}\langle B^\flat(v_2), v_1\rangle = B(v_1, v_2).$$

Hence, it extends to an injective algebra homomorphism,

$$j\colon \mathrm{Cl}(V) \to \mathrm{Cl}(W).$$

Using the covariant spinor module, $\mathrm{Cl}(W)$ is identified with $\mathrm{End}(\wedge(V))$.

PROPOSITION 3.1. *The composition*

$$\mathrm{Cl}(V) \xrightarrow{j} \mathrm{Cl}(W) \xrightarrow{\varrho} \mathrm{End}(\wedge(V))$$

is equal to the standard representation of $\mathrm{Cl}(V)$ *on* $\wedge(V)$*. In particular, the symbol map can be written in terms of the covariant spinor module as*

$$\sigma(x) = \varrho(j(x)).1$$

PROOF. The elements $j(v) = B^\flat(v) \oplus v$ act on $\wedge(V)$ as $\epsilon(v) + \iota(B^\flat(v))$, as required. ☐

If B is non-degenerate, we may interpret the symbol map also in terms of the contravariant spinor module $\wedge(V^*)$. The Clifford action of $j(v)$ is now $\iota(v) + \epsilon(B^\flat(v))$, and we again have $\sigma(x) = \varrho(j(x)).1$. In fact, the isomorphism $V^* \cong V$ given by B defines an isomorphism of the two spinor modules.

3.2. The symbol of elements in the spin group. We will assume that the bilinear form B on V is non-degenerate, and denote by $B^\sharp\colon V^* \to V$ the inverse of $B^\flat$. Let V^- denote the vector space V with bilinear form $-B$. Then the map

$$\kappa\colon V \oplus V^- \to W,\ v_1 \oplus v_2 \mapsto B^\flat(v_1 + v_2) \oplus (v_1 - v_2)$$

is an isomorphism of quadratic vector spaces. Indeed,

$$Q_{B_W}(B^\flat(v_1 + v_2) \oplus (v_1 - v_2)) = B(v_1 + v_2, v_1 - v_2) = Q_B(v_1) - Q_B(v_2).$$

The inverse map reads

$$\kappa^{-1}(\mu \oplus v) = \frac{B^\sharp(\mu) + v}{2} \oplus \frac{B^\sharp(\mu) - v}{2}.$$

In matrix form,

$$\kappa = \begin{pmatrix} B^\flat & B^\flat \\ I & -I \end{pmatrix}, \qquad \kappa^{-1} = \tfrac{1}{2}\begin{pmatrix} B^\sharp & I \\ B^\sharp & -I \end{pmatrix}$$

Conjugation by κ gives a group isomorphism between $\mathrm{O}(V\oplus V^-)$ and $\mathrm{O}(W)$. In particular, $\mathrm{O}(V)\subset \mathrm{O}(V\oplus V^-)$ acts by orthogonal transformations on W. We find, for all $C\in \mathrm{O}(V)$,

$$\kappa\circ\begin{pmatrix} C & 0\\ 0 & I\end{pmatrix}\circ\kappa^{-1}=\tfrac{1}{2}\begin{pmatrix} B^\flat(C+I)B^\sharp & B^\flat(C-I)\\ (C-I)B^\sharp & C+I\end{pmatrix}. \tag{54}$$

If $C+I$ is invertible, we may apply our factorization formula to this expression. We obtain

$$\kappa\circ\begin{pmatrix} C & 0\\ 0 & I\end{pmatrix}\circ\kappa^{-1}=\begin{pmatrix} I & E_2\\ 0 & I\end{pmatrix}\begin{pmatrix} (T^{-1})^* & 0\\ 0 & T\end{pmatrix}\begin{pmatrix} I & 0\\ E_1 & I\end{pmatrix}$$

where

$$E_1=\frac{C-I}{C+I}\circ B^\sharp,\qquad E_2=B^\flat\circ\frac{C-I}{C+I},\qquad T=\frac{C+I}{2}.$$

Note that since the right hand side of this product is in $\mathrm{SO}(W)$, so is the left hand side. We hence see that

$$C\in\mathrm{O}(V),\det(C+I)\neq 0\Rightarrow C\in\mathrm{SO}(V)$$

for any field $\mathbb{K}$ of characteristic 0. (If $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, this fact easily follows from eigenvalue considerations.) We will now consider lifts of $C\in\mathrm{SO}(V)$ to the Clifford group $S\Gamma(V)$, and consider the action of such a lift on the spinor module. This formula will involve the elements $\lambda(E_1),\lambda(E_2)$. For any $X\in\mathfrak{o}(V)$, the element $\lambda(X\circ B^\sharp)\in\wedge^2(V)\subset\wedge^2(W)$, defined using B_W, is related to the element $\lambda(X)\in\wedge^2(V)$, defined using B, by

$$\lambda(X\circ B^\sharp)=2\lambda(X),$$

as follows from the explicit formula (52). Similarly, we have

$$\lambda(B^\flat\circ X)=2\,B^\flat(\lambda(X)).$$

A lift $\widetilde{C}\in S\Gamma(V)$ to the spin group determines a lift

$$\widetilde{T}=\widetilde{\tfrac{C+I}{2}}\in\mathrm{GL}_\Gamma(V)\subset S\Gamma(W).$$

From the known action of the factors in the spinor module, we therefore deduce, using Proposition 3.1:

THEOREM 3.2. *Let $\tilde{C}\in S\Gamma(V)$ be a lift of $C\in\mathrm{SO}(V)$. Then the action of $\tilde{C}$ on the Clifford module $\wedge V$ given by the formula,*

$$\varrho(\tilde{C})=\chi(\widetilde{\tfrac{C+I}{2}})\;\epsilon\big(e^{2\lambda(\frac{C-I}{C+I})}\big)\circ\big(\tfrac{C+I}{2}\big)_*\circ\iota\big(e^{2\lambda(\frac{C-I}{C+I})}\big)$$

where $T_*=(T^{-1})^*$ denotes the action as an algebra homomorphism of $\wedge(V^*)\cong\wedge(V)$. If $\tilde{C}\in\mathrm{Spin}(V)$, the scalar factor may be written

$$\chi(\widetilde{\tfrac{C+I}{2}})=\det{}^{1/2}(\tfrac{C+I}{2}) \tag{55}$$

where the sign of the square root depends on the choice of lift.

Applying this formula to $1\in\wedge(V)$, and using that $\sigma(\tilde{C})=\varrho(\tilde{C}).1$ we find:

PROPOSITION 3.3. *Suppose $\tilde{C} \in S\Gamma(V) \subset \mathrm{Cl}(V)$ is a lift of $C \in \mathrm{SO}(V)$. Then the symbol $\sigma(\tilde{C})$ is given by the formula*

$$\sigma(\tilde{C}) = \chi(\widetilde{\tfrac{C+I}{2}})\ e^{2\lambda(\frac{C-I}{C+I})}.$$

If $\tilde{C} \in \mathrm{Spin}(V)$ the scalar the scalar factor may be written $\det^{1/2}(\frac{C+I}{2})$.

Proposition 3.3 has the following immediate consequence:

COROLLARY 3.4. *Suppose $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{K} = \mathbb{C}$). The pull-back of the function*

$$\mathrm{SO}(V) \to \mathbb{K}, \quad C \mapsto \det(\tfrac{C+I}{2})$$

to $\mathrm{Spin}(V)$ *has a unique smooth (resp. holomorphic) square root, equal to* 1 *at the group unit.*

PROOF. The form degree 0 part $\sigma(\tilde{C})_{[0]}$ of the symbol of $\tilde{C} \in \mathrm{Spin}(V)$ provides such a square root. $\square$

The element

$$\psi_C = \sigma(\tilde{C}) \in \wedge(V^*)$$

is a pure spinor, since it is obtained from the pure spinor $1 \in \wedge(V^*)$ by the action of an element of the spin group. The Lagrangian subspace of W defined by $1 \in \wedge(V^*)$ is V, hence the Lagrangian subspace defined by $\psi_C = \varrho(\tilde{C}).1$ is the image of V under $(\kappa \circ (C \oplus I) \circ \kappa^{-1})V$. That is,

$$F_C = \{((C - I)v,\ (C + I)v) \in W, \quad v \in V\}.$$

A Lagrangian subspace transverse to F_C is given as the image of V^* under $\kappa \circ (C \oplus I) \circ \kappa^{-1}$, that is

$$E_C = \{((C + I)v,\ (C - I)v) \in W, \quad v \in V\}.$$

Any volume element $\Gamma_\wedge \in \det(V^*)$ is a pure spinor defining V^*. Hence a pure spinor defining E_C is

$$\phi_C = \varrho(\tilde{C})\Gamma_\wedge.$$

Note that $(\phi_C, \psi_C) = \Gamma_\wedge$ is non-zero, as required by Theorem 4.7.

3.3. Another factorization. Other types of factorizations of the matrix (54) defined by C lead to different formulas for symbols. We will use the following expression, in which the block-diagonal part is moved all the way to the right:

PROPOSITION 3.5. *Suppose $C \in \mathrm{SO}(V)$ with $\det(C - I) \neq 0$, and let $D\colon V \to V^*$ be skew-adjoint and invertible. Then*

$$\kappa \circ \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \circ \kappa^{-1}$$
$$= \begin{pmatrix} I & 0 \\ E_1 & I \end{pmatrix} \begin{pmatrix} I & D \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ E_2 & I \end{pmatrix} \begin{pmatrix} (T^{-1})^* & 0 \\ 0 & T \end{pmatrix}$$

where

$$E_1 = \frac{C+I}{C-I}\, B^\sharp - D^{-1},$$
$$E_2 = D^{-1}\left(B^\flat\, \frac{C-C^{-1}}{4} D^{-1} - I\right),$$
$$T = D^{-1} B^\flat\, \frac{C-I}{2}.$$

PROOF. The matrix product on the left hand side of the desired equality is given by (54), while the right hand side is, by direct computation,

$$\text{r.h.s.} \;=\begin{pmatrix} (I+DE_2)(T^{-1})^* & DT \\ (E_1+E_1DE_2+E_2)(T^{-1})^* & (I+DE_1)T \end{pmatrix}$$

The two expressions coincide if and only if E_1, E_2, T are as stated in the Proposition. For instance, a comparison of the upper right corners gives $T = D^{-1}\frac{C-I}{2}$. Similarly, one finds E_1, E_2 by comparing the upper left and lower right corners. (One may verify that, with the resulting choices of E_1, E_2, T, the lower left corners match as well.) $\square$

Using the known action of the factors in the spinor module (cf. Section 2.3), we obtain:

COROLLARY 3.6. *Let $\tilde{C} \in S\Gamma(V)$ be a lift of $C \in \mathrm{SO}(V)$. For any choice of D as above, the action of $\tilde{C}$ on $\psi \in \wedge V$ is given by the formula,*

$$\varrho(\tilde{C})\psi = \chi(\tilde{T})\ \iota\big(e^{\lambda(E_1)}\big)\ e^{\lambda(D)}\ \iota\big(e^{\lambda(E_2)}\big)\ T_*\psi$$

Here $\tilde{T}$ is the lift of T determined by the lift $\tilde{C}$. In particular, taking $\psi = 1$ we obtain the following formula for the symbol of elements in the Clifford group:

$$\sigma(\tilde{C}) = \chi(\tilde{T})\ \iota\big(e^{\lambda(E_1)}\big)\ e^{\lambda(D)}.$$

The choice $D = B^\flat\ \frac{C-I}{C+I}$ gives $E_1 = 0$, and we recover our first formula (Proposition 3.3) for $\sigma(\tilde{C})$. In the following Section, we will instead consider the case $C = \exp(A)$ with the choice $D = B^\flat\ A/2$.

3.4. The symbol of elements $\exp(\gamma(A))$**.** Suppose $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and consider exponentials $C = \exp(A)$ for $A \in \mathfrak{o}(V)$. Thus

$$\det(\tfrac{C+I}{2}) = \det(\cosh(\tfrac{A}{2})), \qquad \tfrac{C-I}{C+I} = \tanh(\tfrac{A}{2})$$

By Corollary 3.4 we obtain a smooth square root of the function

$$\mathfrak{o}(V) \to \mathbb{C},\ A \mapsto \det(\cosh(A/2)),$$

equal to 1 at $A = 0$. There is a distinguished lift

$$\tilde{C} = \exp(\gamma(A)) \in \mathrm{Spin}(V),$$

and the formula from Proposition 3.3 now reads:

$$\sigma(e^{\gamma(A)}) = \det{}^{1/2}(\cosh(A/2))\ e^{2\lambda(\tanh(A/2))}. \tag{56}$$

EXAMPLE 3.7. Let us verify that (56) matches with the computation from Example 1.11, in case $V = \mathbb{C}^2$, $\lambda(A) = -\theta/2\ e_1 \wedge e_2$. We had found that

$$\exp(\gamma(A)) = \cos(\theta/2) - \sin(\theta/2)e_1 e_2. \tag{57}$$

On the other hand, $A = \theta J$ where $Je_1 = e_2,\ Je_2 = -e_1$, hence $\exp(A/2) = \cos(\theta/2) + \sin(\theta/2)J$. It follows that

$$\cosh(A/2) = \cos(\theta/2)\,\mathrm{id},$$
$$\tanh(A/2) = \tan(\theta/2)J.$$

This yields $\lambda(\tanh(A/2)) = -\frac{1}{2}\tan(\theta/2)e_1 \wedge e_2$ and

$$\det{}^{1/2}(\cosh(A/2)) = \cos(\theta/2), \quad e^{2\lambda(\tanh(A/2))} = 1 - \tan(\theta/2)e_1 \wedge e_2.$$

Their product is indeed the symbol of (57).

3.5. Clifford exponentials versus exterior algebra exponentials. We first assume $\mathbb{K} = \mathbb{C}$. Let $A \in \mathfrak{o}(V)$, and consider the formula from Section 3.3 for $C = \exp A,\ D = B^\flat \circ \frac{A}{2}$. We obtain

$$E_1 = 2f(A) \circ B^\sharp, \quad E_2 = 2B^\flat \circ g(A), \quad T = j^R(A)$$

with the following functions of $z \in \mathbb{C}$,

$$f(z) = \frac{1}{2}\coth(\frac{z}{2}) - \frac{1}{z}, \quad g(z) = \frac{\sinh(z) - z}{z^2}, \quad j^R(z) = \frac{e^z - 1}{z}.$$

For later use we also define

$$j(z) = \frac{\sinh(z/2)}{z/2}, \quad j^L(z) = \frac{1 - e^{-z}}{z}.$$

Note that g, j^L, j^R, j are entire holomorphic function on $\mathbb{C}$, while f is meromorphic with poles at $2\pi\sqrt{-1}k$ with $k \in \mathbb{Z} - \{0\}$. Since f, g are odd functions, $f(A), g(A)$ are again in $\mathfrak{o}(V)$, while $j(A)^\top = j(A)$ and $j^L(A)^\top = j^R(A)$. Furthermore, $j^L(A), j^R(A), j(A)$ are invertible if and only if A has no eigenvalues of the form $2\pi\sqrt{-1}k$ with $k \in \mathbb{Z} - \{0\}$. The resulting formula for the symbol gives:

THEOREM 3.8. *Suppose $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For all $A \in \mathfrak{o}(V)$ with the property that A has no eigenvalue $2\pi\sqrt{-1}k$ with $k \neq 0$, the symbol of $\exp(\gamma(A)) \in \mathrm{Cl}(V)$ is given by the formula,*

$$\sigma(\exp(\gamma(A)) = \iota(\mathcal{S}(A))\exp(\lambda(A)).$$

where $\mathcal{S}\colon \mathfrak{o}(V) \to \wedge(V)$ is the map

$$\mathcal{S}(A) = \det{}^{1/2}(j(A))\ \exp\big(4\lambda(f(A))\big).$$

Once again, while Proposition 3.5 requires that A is invertible (which never happens if $\dim V$ is odd), the resulting formula holds without this assumption. In case A is invertible we can directly write

$$\mathcal{S}(A) = \det{}^{1/2}\big(\tfrac{\sinh(A/2)}{A/2}\big)\ e^{4\lambda\left(\frac{1}{2}\coth(\frac{A}{2}) - \frac{1}{A}\right)}.$$

As it stands, $\mathcal{S}\colon \mathfrak{o}(V) \to \wedge(V)$ is a meromorphic function, holomorphic on the set of A that do not have eigenvalues of the form $2\pi\sqrt{-1}k$ with $k \in \mathbb{Z} - \{0\}$. We will see below that it is in fact holomorphic everywhere.

EXAMPLE 3.9. We continue the calculations from Example 3.7, where

$$\exp(\gamma(A)) = \cos(\theta/2) - \sin(\theta/2)e_1e_2.$$

We have $\frac{\sinh(A/2)}{A/2} = \frac{\sin(\theta/2)}{\theta/2}I$, hence

$$\det{}^{1/2}\Big(\frac{\sinh(A/2)}{A/2}\Big) = \frac{\sin(\theta/2)}{\theta/2}.$$

On the other hand,

$$f(A) = \frac{1}{2}\coth\frac{A}{2} - \frac{1}{A} = -\big(\frac{1}{2}\cot(\frac{\theta}{2}) - \frac{1}{\theta}\big)J,$$

so

$$4\lambda(f(A)) = (\cot(\frac{\theta}{2}) - \frac{2}{\theta})e_1 \wedge e_2.$$

Hence

$$\mathcal{S}(A) = \frac{\sin(\theta/2)}{\theta/2}\Big(1 + (\cot(\frac{\theta}{2}) - \frac{2}{\theta})e_1 \wedge e_2\Big).$$

Note that $\mathcal{S}(A)$ has no poles. Contracting with $\exp(\lambda(A)) = 1 - \frac{\theta}{2}e_1 \wedge e_2$ we find,

$$\iota(\mathcal{S}(A))\exp(\lambda(A)) = \cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})e_1 \wedge e_2 = \sigma(\exp(\gamma(A))),$$

as desired.

3.6. The symbol of elements $\exp(\gamma(A) - \sum_i e_i\tau^i)$**.** Theorem 3.8 has a useful generalization, allowing linear terms. Let P be a vector space of odd "parameters". Let e_i be a basis of V, and consider expressions $e_i \otimes \tau^i \in V \otimes P$ with $\tau^i \in P$. Note that we can view that parameters τ^i as the components $\tau(e^i)$ of a linear map $\tau\colon V \to P$. The following Theorem shows that the same element $\mathcal{S}(A) \in \wedge(V)$ as before relates the exponentials of elements

$$\lambda(A) - \sum_i e_i\tau^i \in \wedge(V) \otimes \wedge(P),$$
$$\gamma(A) - \sum_i e_i\tau^i \in \mathrm{Cl}(V) \otimes \wedge(P).$$

Quite remarkably, it is not necessary to introduce a τ-dependence into the definition of $\mathcal{S}(A)$.

THEOREM 3.10. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. With $\mathcal{S}(A)$ as above,*

$$e^{\gamma(A) - \sum_i e_i\tau^i} = q\Big(\iota(\mathcal{S}(A))e^{\lambda(A) - \sum_i e_i\tau^i}\Big).$$

PROOF. Think of $\mathrm{Cl}(V) \otimes \wedge(P) = \mathrm{Cl}(V \oplus P)$ as the Clifford algebra for the degenerate bilinear form $B \oplus 0$. Pick an arbitrary non-degenerate symmetric bilinear form B_P on P, and consider the bilinear form $B \oplus \epsilon B_P$ on $V \oplus P$. Then $\lambda(A) - \sum_i e_i \tau^i = \lambda(\tilde{A}_\epsilon)$ with

$$\tilde{A}_\epsilon = \begin{pmatrix} A & -2\epsilon\tau^\top \\ 2\tau & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 2\tau & 0 \end{pmatrix} + O(\epsilon),$$

where $O(\epsilon)$ denotes a term that goes to 0 for $\epsilon \to 0$. By induction, it follows that the powers of $\tilde{A}_\epsilon$ have the form

$$\tilde{A}_\epsilon^m = \begin{pmatrix} A^m & 0 \\ 2\tau A^{m-1} & 0 \end{pmatrix} + O(\epsilon).$$

Hence $f(\tilde{A}_\epsilon) \in \mathrm{End}(V \oplus P)$ is given by

$$f(\tilde{A}_\epsilon) = \begin{pmatrix} f(A) & 0 \\ Q & 0 \end{pmatrix} + O(\epsilon),$$

with $Q = \tau f(A) A^{-1}$. The skew-adjoint map $V \oplus P \to V^* \oplus P^*$ defined by $\lambda(f(\tilde{A}_\epsilon)) \in \wedge^2(V^* \oplus P^*)$ is the composition

$$\begin{pmatrix} B_V^\flat & 0 \\ 0 & \epsilon B_P^\flat \end{pmatrix} \circ f(\tilde{A}_\epsilon) = \begin{pmatrix} B_V^\flat \circ f(A) & 0 \\ 0 & 0 \end{pmatrix} + O(\epsilon).$$

This shows

$$\lambda(f(\tilde{A}_\epsilon)) = \lambda(f(A)) + O(\epsilon).$$

Similarly,

$$\det(j(\tilde{A}_\epsilon)) = \det(j(A)) + O(\epsilon),$$

since only the block diagonal term contributes. The Theorem now follows by letting $\epsilon \to 0$ in our general formula,

$$\exp(\gamma(\tilde{A}_\epsilon)) = \iota(\mathcal{S}(\tilde{A}_\epsilon)) \exp(\lambda(\tilde{A}_\epsilon)). \qquad \square$$

EXAMPLE 3.11. In the 2-dimensional setting of Examples 3.7 and 3.9, consider exponentials of the form

$$e^{\gamma(A) - \sum_{i=1}^2 e_i \tau^i} = e^{-\frac{\theta}{2} e_1 e_2 - \sum_{i=1}^2 e_i \tau^i} = \sum_{m=0}^\infty \frac{(-1)^m}{m!} \big(\frac{\theta}{2} e_1 e_2 + \sum_{i=1}^2 e_i \tau^i\big)^m.$$

Using that $e_1 e_2$ anti-commutes with $\sum_i e_i \tau^i$, the sum becomes

$$\sum_{m=0}^\infty \frac{(-1)^m}{m!} \big(\frac{\theta}{2}\big)^m (e_1 e_2)^m + \sum_{m=1}^\infty \frac{(-1)^m}{m!} a_m \big(\sum_{i=1}^2 e_i \tau^i\big) \big(\frac{\theta}{2} e_1 e_2\big)^{m-1}$$
$$+ \sum_{m=2}^\infty \frac{(-1)^m}{m!} b_m \big(\sum_{i=1}^2 e_i \tau^i\big)^2 \big(\frac{\theta}{2} e_1 e_2\big)^{m-2}$$

where $a_m = \sum_{j=0}^{m-1} (-1)^j = \frac{1}{2}(1 + (-1)^m)$ is 1 if m is odd, 0 otherwise, while

$$b_m = \sum_{j,j' \geq 0, j+j' \leq m-2} (-1)^{j+j'} = \tfrac{1}{2}\big((-1)^m m + \frac{1}{2}(1 - (-1)^m)\big).$$

With $(e_1e_2)^{2k} = (-1)^k$, the result can as before be expressed in terms of trigonometric functions:

$$\begin{aligned} e^{\gamma(A)-\sum_{i=1}^2 e_i\tau^i} &= \cos(\theta/2) - \sin(\theta/2)e_1e_2 - \frac{\sin(\theta/2)}{\theta/2}\sum_{i=1}^2 e_i\tau^i \\ &+ \Big(\frac{\sin(\theta/2)}{(\theta/2)^2} - \frac{\cos(\theta/2)}{\theta/2}\Big)\tau_1\tau_2 - \frac{\sin(\theta/2)}{\theta/2}e_1e_2\tau_1\tau_2. \end{aligned}$$

It is straightforward to verify that this formula coincides with $q\circ\iota(\mathcal{S}(A))$ applied to

$$e^{\lambda(A)-\sum_{i=1}^2 e_i\wedge\tau^i} = 1 - \frac{\theta}{2}e_1\wedge e_2 - \sum_{i=1}^2 e_i\tau^i + e_1\wedge e_2\wedge\tau_1\wedge\tau_2.$$

3.7. The function $A\mapsto\mathcal{S}(A)$. Until now, the function $\mathcal{S}$ was defined on the set of $A\in\mathfrak{o}(V)$ such that A has no non-zero eigenvalues in $2\pi\sqrt{-1}\mathbb{Z}$. We can now show that the function $\mathcal{S}$ extends to an analyic function on all of $\mathfrak{o}(V)$. In particular, the formulas established above hold on all of $\mathfrak{o}(V)$.

THEOREM 3.12. *Let $\mathbb{K} = R$ or $\mathbb{C}$. The function $A\mapsto\mathcal{S}(A)$ extends to an analytic function $\mathfrak{o}(V)\to\wedge(V)$. In particular, its degree zero part*

$$A\mapsto \det{}^{1/2}(j(A)) = \det{}^{1/2}\big(\tfrac{\sinh(A/2)}{A/2}\big)$$

is a well-defined analytic function $\mathfrak{o}(V)\to\mathbb{K}$.

PROOF. Let P,τ^i be as in Theorem 3.10. We assume that the τ^i are a basis of the parameter space P (e.g. we may take $P = V^*$, with τ^i the dual basis of e_i). Then $\exp(\lambda(A) - \sum_i e_i\tau^i)$ has a non-vanishing part of top degree $2\dim V$. By the Lemma below, there exists an analytic function $\mathcal{S}'\colon\mathfrak{o}(V)\to\wedge^2(V\oplus P^*)$ satisfying

$$\iota(\mathcal{S}'(A))\exp(\lambda(A) - \sum_i e_i\tau^i) = q^{-1}(\exp(\gamma(A) - \sum_i e_i\tau^i)).$$

By uniqueness, this function coincides with the function $\mathcal{S}(A)$ defined above. (Thus, it actually takes values in $\wedge(V)$.) $\square$

LEMMA 3.13. *Let E be a vector space, $\chi,\psi\in\wedge(E)$, and suppose that the top degree part $\chi_{[\mathrm{top}]}\in\det(E)$ is non-zero. Then there is a unique solution $\phi\in\wedge(E^*)$ of the equation*

$$\psi = \iota(\phi)\chi.$$

If χ,ψ depend analytically on parameters, then so does the solution ϕ.

PROOF. Fix a generator $\Gamma_\wedge\in\det(E^*)$. Then the desired equation $\psi = \iota(\phi)\chi$ is equivalent to

$$\iota(\psi)\Gamma_\wedge = \phi\wedge\iota(\chi)\Gamma_\wedge.$$

Since $\chi_{[\dim E]}\neq 0$, we have $(\iota(\chi)\Gamma_\wedge)_{[0]}\neq 0$, i.e. $\iota(\chi)\Gamma_\wedge$ is invertible. Thus

$$\phi = (\iota(\psi)\Gamma_\wedge)\wedge(\iota(\chi)\Gamma_\wedge)^{-1}$$

This shows existence and uniqueness, and also implies the statements regarding dependence on parameters. □

4. Volume forms on conjugacy classes

As an application of some the techniques developed here, we will prove the following fact:

> *the conjugacy classes of any connected, simply connected semi-simple (real or complex) Lie group carry distinguished volume forms.*

In fact, it suffices to assume that the Lie algebra of G carries an invariant non-degenerate symmetric bilinear form B - in the semi-simple case this can be taken to be the Killing form. Also, the assumption that G is connected and simply conected can be relaxed (see below).

We begin with a real or complex Lie group G, together with a G-invariant symmetric bilinear form B on $\mathfrak{g}$. Let $\theta^L,\ \theta^R \in \Omega^1(G,\mathfrak{g})$ be the left-, right-invariant Maurer-Cartan forms. That is, if $\xi^L,\ \xi^R \in \mathfrak{X}(G)$ are the left-, right-invariant vector fields on G, equal to ξ at the group unit, then $\iota(\xi^L)\theta^L = \xi = \iota(\xi^R)\theta^R$. For $\xi \in \mathfrak{g}$ define two sections of the bundle $T^*G \oplus TG$:

$$e(\xi) = B(\theta^L + \theta^R, \xi) \oplus (\xi^L - \xi^R),$$
$$f(\xi) = B(\theta^L - \theta^R, \xi) \oplus (\xi^L + \xi^R).$$

Proposition 4.1. *The sections $e(\xi)$, $f(\xi)$ for $\xi \in \mathfrak{g}$ span transverse Lagrangian sub-bundles $E, F \subset T^*G \oplus TG$, respectively. One has*

$$\langle e(\xi_1), e(\xi_2)\rangle = 0, \quad \langle f(\xi_1), f(\xi_2)\rangle = 0, \quad \langle e(\xi_1), f(\xi_2)\rangle = 2B(\xi_1, \xi_2).$$

Proof. Use left trivalization of the tangent bundle to identify $TG = G \times \mathfrak{g}$ and $T^*G = G \times \mathfrak{g}^*$. The left trivialization takes $B(\theta^L, \xi),\ \xi^L$ to the constant sections $g \mapsto B^\flat(\xi),\ \xi$ and $B(\theta^R, \xi), \xi^R$ to the sections $g \mapsto B^\flat(\mathrm{Ad}(g^{-1})\xi),\ \mathrm{Ad}(g^{-1})\xi$. Thus, in terms of left trivialization,

$$e(\xi)|_g = B^\flat((1 + \mathrm{Ad}(g^{-1}))\xi) \oplus (1 - \mathrm{Ad}(g^{-1})\xi),$$
$$f(\xi)|_g = B^\flat((1 - \mathrm{Ad}(g^{-1}))\xi) \oplus (1 + \mathrm{Ad}(g^{-1})\xi).$$

Let $\overline{TG}$ denote TG with the opposite bilinear form, and define the bundle map

$$\kappa\colon TG \oplus \overline{TG} \to T^*G \oplus TG,\ v_1 \oplus v_2 \mapsto B^\flat(v_1 + v_2) \oplus (v_1 - v_2)$$

as in Section **??**. This is an isometry, and we observe that

$$\kappa^{-1}(e(\xi)) = \xi \oplus \mathrm{Ad}(g^{-1})\xi,\ \ \kappa^{-1}(f(\xi)) = \xi \oplus (-\mathrm{Ad}(g^{-1})\xi). \tag{58}$$

That is, $\kappa^{-1}(E)|_g$ is the graph of the isometry $\mathrm{Ad}(g^{-1})$, while $\kappa^{-1}(F)|_g$ is the graph of the isometry $-\mathrm{Ad}(g^{-1})$. These are transverse Lagrangian subspaces, hence so is their image under κ. The formulas for the inner products are immediate from (58). □

The TG-components of the sections $e(\xi) \in \Gamma(T^*G \oplus TG)$ are the generating vector fields $\xi_G = \xi^L - \xi^R \in \mathfrak{X}(G)$ for the conjugation action on G. Suppose $\mathcal{C} \subset G$ is a conjugacy class, i.e. an orbit of the conjugation action, and define $E_{\mathcal{C}} = T\mathcal{C} \oplus T^*\mathcal{C}$ to be the span of the sections

$$e_{\mathcal{C}}(\xi) = \iota_{\mathcal{C}}^* B(\theta^L + \theta^R, \xi) \oplus \xi_{\mathcal{C}}$$

where $\iota_{\mathcal{C}} \colon \mathcal{C} \hookrightarrow G$ is the inclusion. Then $E_{\mathcal{C}}$ is isotropic (since $\langle e_{\mathcal{C}}(\xi_1), e_{\mathcal{C}}(\xi_2)\rangle = \langle e(\xi_1), e(\xi_2)\rangle|_{\mathcal{C}} = 0$). For dimension reasons, it is in fact Lagrangian. Since the projection of $E_{\mathcal{C}}$ to $T\mathcal{C}$ is a bijection, $E_{\mathcal{C}}$ is the graph of a 2-form $-\omega_{\mathcal{C}}$ where

$$\iota(\xi_{\mathcal{C}})\omega_{\mathcal{C}} = -i_{\mathcal{C}}^* B(\theta^L + \theta^R, \xi)$$

Explitly,

$$\omega_{\mathcal{C}}(\xi_{\mathcal{C}}|_g, \xi'_{\mathcal{C}}|_g) = B(\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}}\xi, \xi').$$

In terms of the contravariant spinor module $\wedge T^*\mathcal{C}$ over $\mathrm{Cl}(T^*\mathcal{C} \oplus T\mathcal{C})$, it follows that $E_{\mathcal{C}}$ is the Lagrangian sub-bundle spanned by the pure spinor $\phi_{\mathcal{C}} = \exp(-\omega_{\mathcal{C}}) \in \Omega(\mathcal{C})$.

By definition of $E_{\mathcal{C}}$, we have $E|_{\iota(g)} = (T_g\iota_{\mathcal{C}})_! E_{\mathcal{C}}|_g$ for all $g \in \mathcal{C}$, in the notation of Section 1. By Proposition 1.3, the Lagrangian sub-bundle $F_{\mathcal{C}} \subset T^*\mathcal{C} \oplus T\mathcal{C}$ given by $F_{\mathcal{C}}|_g = (T_g\iota_{\mathcal{C}})^! F|_{\iota(g)}$ is transverse to $E_{\mathcal{C}}$. Suppose we are given a pure spinor $\psi \in \Omega(G)$ defining $F_{\mathcal{C}}$. Then the pull-back $\psi_{\mathcal{C}} = \iota_{\mathcal{C}}^*\psi$ is a pure spinor defining $F_{\mathcal{C}}$. By §3, Theorem 4.7, the transversality of $E_{\mathcal{C}}$, $F_{\mathcal{C}}$ is equivalent to the non-dgeneracy of the pairing between $\phi_{\mathcal{C}}$ and $\psi_{\mathcal{C}}$. That is, $(\phi_{\mathcal{C}}, \psi_{\mathcal{C}}) = (\exp(\omega_{\mathcal{C}}) \wedge \iota_{\mathcal{C}}^*\psi$ is a volume form on $\mathcal{C}$.

We will give a construction of ψ using the spin representation. This will require an additional assumption: We assume that we are given a lift

$$\widetilde{\mathrm{Ad}} \colon G \to \mathrm{Pin}(\mathfrak{g})$$

of the adjoint action $\mathrm{Ad} \colon G \to \mathrm{O}(\mathfrak{g})$. If G is connected and simply connected, the lift is automatic.

Recall the description of F given in the proof of Proposition 4.1, presenting F as the image of $\{(\mathrm{Ad}(g)\xi, -\xi)|\ \xi \in \mathfrak{g}\} \subset T\mathcal{C} \oplus \overline{T\mathcal{C}}$ under the isometry κ. The image of the anti-diagonal $\{(\xi, -\xi)|\ \xi \in \mathfrak{g}\}$ under κ is T^*G, which is defined by the pure spinor $1 \in \Gamma(\wedge T^*G) = \Omega(G)$. Thus

$$\psi_g = \varrho(\widetilde{\mathrm{Ad}}(g)).1$$

is a pure spinor representing $F|_g$. By Proposition 3.1, this is just the symbol of the element $\widetilde{\mathrm{Ad}}(g) \in \mathrm{Pin}(\mathfrak{g}) \subset \mathrm{Cl}(\mathfrak{g})$:

$$\psi_g = \sigma(\widetilde{\mathrm{Ad}}(g)).$$

Note that ψ is invariant under the adjoint action of G on itself. To summarize:

THEOREM 4.2 (Volume forms on conjugacy classes). [**3, 52**] *Let G be a Lie group, whose Lie algebra carries an invariant non-degenerate symmetric bilinear form B. Assume that the adjoint action admits a lift $G \to \mathrm{Pin}(\mathfrak{g})$,* *and let $\psi \in \Omega(G)$ be the resulting pure spinor. For any conjugacy class $\mathcal{C} \subset G$, let $\omega_{\mathcal{C}}$ be the 2-form defined above. Then the top degree part of the wedge product*

$$\exp(\omega_{\mathcal{C}}) \wedge i_{\mathcal{C}}^* \psi$$

defines an $\mathrm{Ad}(G)$*-invariant volume form on* $\mathcal{C}$.

Proposition 3.3 (applied to $C = \mathrm{Ad}(g)$) gives an explicit formula for ψ. Left trivialization of TG identifies the map $g \mapsto \lambda(\frac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1}) \in \wedge^2 \mathfrak{g}^*$ with a 2-form on G, defined over the subset of G where Ad_g has no eigenvalue equal to -1. In terms of Maurer-Cartan forms the 2-form reads

$$-\frac{1}{4} B(\tfrac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1} \theta^L, \theta^L)$$

We obtain:

Proposition 4.3. *Over the set of $g \in G$ where Ad_g has no eigenvalue equal to -1, the pure spinor $\psi \in \Omega(G)$ is given by the formula*

$$\psi = \det{}^{1/2}(\tfrac{\mathrm{Ad}_g + 1}{2}) \exp(-\tfrac{1}{4} B(\tfrac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1} \theta^L, \theta^L)). \tag{59}$$

Here the sign of the square root depends on the choice of lift $\widetilde{\mathrm{Ad}}$. If G is connected, the set of $g \in G$ such that $\det(\mathrm{Ad}_g + I) = 0$ is open and dense. On the other hand, if $\det(\mathrm{Ad}_g) = -1$ so that $\mathrm{Ad}_g \in \mathrm{O}(\mathfrak{g}) - \mathrm{SO}(\mathfrak{g})$, Ad_g always has -1 as an eigenvalue.

Let us make a few comments on the volume form on $\mathcal{C}$.

Remarks 4.4.
(1) Theorem 4.2 shows in particular that, under the given assumptions, all conjugacy classes in G have a natural orientation. The simplest example of a non-orientable conjugacy class of a non-simply connected group is $\mathcal{C} = \mathbb{R}P(2) \subset \mathrm{SO}(3)$ (the conjugacy class of rotations by π).
(2) If G is connected, the map to $\mathrm{Pin}(\mathfrak{g})$ necessarily takes values in $\mathrm{Spin}(\mathfrak{g})$, and hence the resulting form ψ is *even*. According to the Theorem, the conjugacy classes in G must all be even-dimensional. The simplest example of an odd-dimensional conjugacy class of a disconnected Lie group is $\mathcal{C} = S^1 \subset \mathrm{O}(2)$, the conjugacy class of 2-dimensional reflections.
(3) The adjoint action always lifts after passage to a double cover $\tilde{G}$, if necessary. The volume forms on the conjugacy classes in $\tilde{G}$ determine in particular invariant measures, and these descend to the conjugacy classes in G. Thus, given the invariant metric B on $\mathfrak{g}$, all conjugacy classes in G carry distinguished invariant measures. Note that conjugacy classes in a general Lie group G need not admit invariant measures. An example is the group G generated by translations and dilations of the real line $\mathbb{R}$. The generic conjugacy classes are diffeomorphic to $\mathbb{R}$ with this action, and hence do not carry invariant measures. In this case, $\mathfrak{g}$ does not admit an invariant metric.

(4) If G is semi-simple, there is a distinguished B given by the Killing form. Hence in that case the volume forms on the conjugacy classes of G (assuming e.g. that G is simply connected) are completely canonical.
(5) The volume forms on conjugacy classes are analogous to the Liouville volume forms on coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$. The latter are given by $(\exp \omega_{\mathcal{O}})_{[\mathrm{top}]}$, using the Kirillov-Kostant-Souriau symplectic structure $\omega_{\mathcal{O}}$.

For further developments of the theory outlined here, see [**3**].

CHAPTER 5

Enveloping algebras

Enveloping algebras define a functor $\mathfrak{g} \mapsto U(\mathfrak{g})$ from the category of Lie algebra to the category of associative unital algebras, in such a way that representation of $\mathfrak{g}$ on a vector space V are equivalent to an algebra representation of $U(\mathfrak{g})$ on V. A fundamental result in the theory of enveloping algebras is the Poincar'e-Birkhoff-Witt theorem, which (in one of its incarnations) states that a natural 'quantization map' from the symmetric algebra $S(\mathfrak{g})$ into $U(\mathfrak{g})$ is an isomorphism of vector spaces. One of the goals of this chapter is to present a proof of this result, due to Petracci, which is similar in spirit to the proof that the quantization map for Clifford algebras is an isomorphism. Throughout this chapter, $\mathbb{K}$ denotes a field of characteristic 0. The vector spaces E and Lie algebras $\mathfrak{g}$ considered in this Chapter may be infinite-dimensional unless stated otherwise.

1. The universal enveloping algebra

1.1. Construction. For any Lie algebra $\mathfrak{g}$, one defines the *universal enveloping algebra* $U(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{I}$ as the quotient of the tensor algebra by the two-sided ideal $\mathcal{I}$ generated by elements of the form

$$\xi \otimes \zeta - \zeta \otimes \xi - [\xi, \zeta].$$

Equivalently, the universal enveloping algebra is generated by elements $\xi \in \mathfrak{g}$ subject to relations $\xi\zeta - \zeta\xi = [\xi, \zeta]$. Since $\mathcal{I}(\mathfrak{g})$ is a filtered ideal in $T(\mathfrak{g})$, with $\mathcal{I} \cap \mathbb{K} = 0$, it follows that $U(\mathfrak{g})$ is a filtered algebra. The construction of the enveloping algebra $U(\mathfrak{g})$ from a Lie algebra $\mathfrak{g}$ is functorial: Any Lie algebra homomorphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$ induces a morphisms of filtered algebras $U(\mathfrak{g}_1) \to U(\mathfrak{g}_2)$, with the appropriate property under composition.

The inclusion $\mathfrak{g} \to T(\mathfrak{g})$ descends to a Lie algebra homomorphism

$$j \colon \mathfrak{g} \to U(\mathfrak{g}),$$

where the bracket on $U(\mathfrak{g})$ is the commutator. As a consequence of the Poincaré-Birkhoff-Witt theorem, to be discussed below, this map is injective. We will usually denote the image $j(\xi)$ in the enveloping algebra simply by ξ, although strictly speaking this is only justified once the Poincaré-Birkhoff-Witt theorem is proved.

1.2. Universal property.

THEOREM 1.1 (Universal property). *If $\mathcal{A}$ is an associative algebra, and $f\colon \mathfrak{g} \to \mathcal{A}$ is a homomorphism of Lie algebras, then there is a unique morphism of algebras $f_U\colon U(\mathfrak{g}) \to \mathcal{A}$ such that $f = f_U \circ j$.*

PROOF. The map f extends to an algebra homomorphism $T(\mathfrak{g}) \to \mathcal{A}$. This algebra homomorphism vanishes on the ideal $\mathcal{I}$, and hence descends to an algebra homomorphism $f_U\colon U(\mathfrak{g}) \to \mathcal{A}$ with the desired property. This extension is unique, since $j(\mathfrak{g})$ generates $U(\mathfrak{g})$ as an algebra. $\square$

By the universal property, any module over the Lie algebra $\mathfrak{g}$ becomes a module over the algebra $U(\mathfrak{g})$.

REMARK 1.2. If $\dim \mathfrak{g} < \infty$, the injectivity of the map $j\colon \mathfrak{g} \to U(\mathfrak{g})$ can also be obtained as a consequence of *Ado's theorem* that any such Lie algebra has a faithful finite-dimensional representation $f\colon \mathfrak{g} \to \mathrm{End}(V)$: Faithfulness means that f is injective, and since $f = f_U \circ j$ it follows that j is injective also.

If $\mathfrak{g}_1, \mathfrak{g}_2$ are two Lie algebras one has an isomorphism of filtered algebras

$$U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$$

(the tensor product on the right satisfies the universal property of the enveloping algebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$). A detailed proof may be found e.g. in [**41**, Chapter V.2]

1.3. Augmentation map, anti-automorphism. The projection $\mathfrak{g} \to 0$ is a Lie algebra homomorphism inducing the *augmentation* $U(\mathfrak{g}) \to \mathbb{K}$. Its kernel is called the augmentation ideal, and is denoted $U^+(\mathfrak{g})$. By contrast, Clifford algebras do not, in general, admit augmentation maps that are (super) algebra homomorphisms.

1.4. Anti-automorphism. The map $\xi \mapsto -\xi$ is an anti-automorphism of the Lie algebra $\mathfrak{g}$, i.e. it preserves the bracket up to a sign. Define an algebra anti-automorphism of $T(\mathfrak{g})$ by $\xi_1 \otimes \cdots \otimes \xi_r \mapsto (-1)^r \xi_r \otimes \cdots \otimes \xi_1$. This preserves the ideal $\mathcal{I}$, and therefore descends to an anti-automorphism of $U(\mathfrak{g})$, denoted s. That is, $\mathsf{s}(\xi_1 \cdots \xi_k) = (-1)^k \xi_k \cdots \xi_1$.

1.5. Derivations. The functoriality of the construction of $U(\mathfrak{g})$ shows in particular that any Lie algebra automorphism of $\mathfrak{g}$ extends uniquely to an algebra automorphism of $U(\mathfrak{g})$. Similarly, if D is a Lie algebra derivation of $\mathfrak{g}$, then the derivation of $T(\mathfrak{g})$ extending D preserves the ideal $\mathcal{I}$, hence it descends to an algebra derivation of $U(\mathfrak{g})$. Thus

$$D(\xi_1 \cdots \xi_k) = \sum_{i=1}^{k} \xi_1 \cdots \xi_{i-1} (D\xi_i) \xi_{i+1} \cdots \xi_k.$$

1.6. Modules over $U(\mathfrak{g})$. A module over $U(\mathfrak{g})$ is a vector space E together with an algebra homomorphism $U(\mathfrak{g}) \to \mathrm{End}(E)$. The universal property of the enveloping algebra shows that such a module structure is equivalent to a Lie algebra representation of $\mathfrak{g}$ on E. The left-action of the enveloping algebra on itself corresponds to the *left regular representation* $\varrho^L(\xi)x = \xi x$. There is also a *right regular representation* $\varrho^R(\xi)x = -x\xi$. The two actions commute, and the diagonal action is the *adjoint representation* $\mathrm{ad}(\xi)x = \xi x - x\xi = [\xi, x]$. An element x lies in the center of $U(\mathfrak{g})$ if and only if it commutes with all generators ξ. That is, it consists exactly of the invariants for the adjoint action:

$$\mathrm{Cent}(U(\mathfrak{g})) = U(\mathfrak{g})^{\mathfrak{g}}.$$

1.7. Unitary representations. Suppose $\mathfrak{g}$ is a real Lie algebra, and let $\mathfrak{g}^{\mathbb{C}}$ be its complexification. The enveloping algebra $U(\mathfrak{g}^{\mathbb{C}})$ carries a unique conjugate linear automorphism $x \mapsto \overline{x}$ extending the complex conjugation map on $\mathfrak{g}^{\mathbb{C}}$. Define a conjugate linear anti-automorphism

$$*\colon U(\mathfrak{g}^{\mathbb{C}}) \to U(\mathfrak{g}^{\mathbb{C}}), \quad x \mapsto x^* = \mathsf{s}(\overline{x}).$$

A unitary representation of $\mathfrak{g}$ on a Hermitian vector space E is a Lie algebra homomorphism $\varrho\colon \mathfrak{g} \to \mathrm{End}(E)$ such that the elements of $\mathfrak{g}$ are represented as skew-adjoint operators. That is, $\varrho(\xi^*) = \varrho(\xi)^*$. Equivalently, it is a $*$-homomorphism $U(\mathfrak{g}^{\mathbb{C}}) \to \mathrm{End}(E)$.

1.8. Graded or filtered Lie (super) algebras. If $\mathfrak{g}$ is a graded (resp. filtered) Lie algebra, then the tensor algebra $T(\mathfrak{g})$ is a graded (resp. filtered) algebra, in such a way that the inclusion of $\mathfrak{g}$ is a morphism. Furthermore, the ideal ideal $\mathcal{I}$ defining the enveloping algebra is a graded (resp. filtered) subspace, and hence the enveloping algebra $U(\mathfrak{g})$ inherits a grading (resp. filtration). Put differently, this *internal grading* (resp. *internal filtration*) is defined by the condition that the inclusion $j\colon \mathfrak{g} \to U(\mathfrak{g})$ preserves degrees. The filtration from the construction in 1.1 will be called the *external filtration.* The *total filtration* degree is the sum of the internal and external filtration degrees. The total filtration degree is such that the map $j\colon \mathfrak{g} \to U(\mathfrak{g})$ defines a morphism of filtered spaces, $\mathfrak{g}[-1] \to U(\mathfrak{g})$. Given a filtered Lie algebra $\mathfrak{g}$, the same Lie algebra with shifted filtration $\mathfrak{g}[-1]$ is again a filtered Lie algebra, and the total filtration for $U(\mathfrak{g})$ agrees with the internal filtration for $U(\mathfrak{g}[-1])$.

If $\mathfrak{g}$ is a Lie super algebra, one defines the enveloping algebra $U(\mathfrak{g})$ as a quotient of the tensor algebra by the ideal generated by elements

$$\xi \otimes \zeta - (-1)^{|\zeta||\xi|}\zeta \otimes \xi - [\xi, \zeta].$$

Then $U(\mathfrak{g})$ becomes a superalgebra, in such a way that $j\colon \mathfrak{g} \to U(\mathfrak{g})$ is a morphism of super spaces. If $\mathfrak{g}$ is a graded (resp. filtered) Lie super algebra, then $U(\mathfrak{g})$ becomes a graded (resp. filtered) super algebra, relative to the *internal* grading (resp. filtration) defined by the condition that j is a morphism of graded (resp. filtered) super vector spaces. It is also a filtered super algebra relative to the *total* filtration, defined by the condition that j defines a morphism of filtered super spaces $\mathfrak{g}[-2] \to U(\mathfrak{g})$. (The degree shift by 2 is dictated by the super-sign convention: Recall that to view graded, filtered vector spaces as graded, filtered super spaces, one doubles the degree.) Given a filtered Lie super algebra $\mathfrak{g}$, the same Lie algebra with shifted filtration $\mathfrak{g}[-2]$ is again a filtered Lie super algebra, and the total filtration for $U(\mathfrak{g})$ agrees with the internal filtration for $U(\mathfrak{g}[-2])$.

1.9. Further remarks.

(1) Given a central extension

$$0 \to \mathbb{K}\mathsf{c} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

of a Lie algebra $\mathfrak{g}$, one can define *level* r *enveloping algebras*

$$U_r(\widehat{\mathfrak{g}}) := U(\widehat{\mathfrak{g}})/ < \mathsf{c} - r >, \quad r \in \mathbb{K}$$

specializing to $U(\mathfrak{g})$ for $r = 0$. A module over $U_r(\widehat{\mathfrak{g}})$ is given by a $\widehat{\mathfrak{g}}$-representation such that the central element c acts as multiplication by r. Again, this construction generalizes to central extensions of graded or filtered super Lie algebras.

(2) Suppose V is a vector space, equipped with a symmetric bilinear form B. Define a graded super Lie algebra

$$\mathbb{K}[2] \oplus V[1],$$

where $\mathbb{K}[2]$ is the 1-dimensional space spanned by a central element c of degree -2, and where $[v, w] = 2B(v, w)\mathsf{c}$ for $v, w \in V[1]$. Shifting degree by 2, this becomes a filtered super Lie algebra $\mathbb{K} \oplus V[-1]$, for which c now has degree 0. Its level 1 enveloping algebra is the Clifford algebra,

$$\mathrm{Cl}(V; B) = U_1(\mathbb{K} \oplus V[-1]);$$

here the filtration on the Clifford algebra comes from the internal filtration of the enveloping algebra.

2. The Poincaré-Birkhoff-Witt theorem

The Poincaré-Birkhoff-Witt theorem appears in several equivalent versions. The first version is based on the following observation.

LEMMA 2.1. *For any permutation σ of $\{1, \dots, k\}$ and any $\xi_j \in \mathfrak{g}$,*

$$\xi_1 \cdots \xi_k - \xi_{\sigma(1)} \cdots \xi_{\sigma(k)} \in U^{(k-1)}(\mathfrak{g}).$$

PROOF. For transpositions of two adjacent elements, this is clear from the definition of the enveloping algebra. The general case follows since such transpositions generate the symmetric group. □

It follows that the commutator or two elements of filtration degree k, l has filtration degree $k+l-1$. Hence, the associated graded algebra $\mathrm{gr}(U(\mathfrak{g}))$ is commutative (in the usual, ungraded sense) and the inclusion of $\mathfrak{g}$ extends to an algebra homomorphism

$$S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g})). \tag{60}$$

THEOREM 2.2 (Poincaré-Birkhoff-Witt, version I). *The homomorphism* $S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ *is an algebra isomorphism.*

For the second version, let $\{e_i,\, i \in P\}$ be a basis of $\mathfrak{g}$, with a totally ordered index set P. Using the Lemma, one shows that $U(\mathfrak{g})$ is already spanned by elements of the form $e_{i_1} \cdots e_{i_k}$ where $i_1 \le \cdots i_k$. Since the corresponding elements in $S(\mathfrak{g})$ are clearly a basis of $S(\mathfrak{g})$ we obtain a surjective linear map

$$S(\mathfrak{g}) \to U(\mathfrak{g}),\ e_{i_1} \cdots e_{i_k} \mapsto e_{i_1} \cdots e_{i_k}. \tag{61}$$

THEOREM 2.3 (Poincaré-Birkhoff-Witt, version II). *The elements*

$$\{e_{i_1} \cdots e_{i_k} |\ i_1 \le \cdots \le i_k\}$$

form a basis of $U(\mathfrak{g})$.

Equivalently, the map (61) is an isomorphism. Since a map of $\mathbb{Z}_{\ge 0}$-filtered vector spaces is an isomorphism if and only if the associated graded map is an isomorphism, and since the associated graded map to (61) is (60), the equivalence of versions I,II is clear. A different lift of (60) is given by *symmetrization*,

$$\mathrm{sym}\colon S(\mathfrak{g}) \to U(\mathfrak{g}),\quad \xi_1 \cdots \xi_k \mapsto \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \xi_{s(1)} \cdots \xi_{s(k)}$$

It may be characterized as the unique linear map such that

$$\mathrm{sym}(\xi^k) = \xi^k$$

for all $\xi \in \mathfrak{g}$ and all k, where on the left hand side the kth power $\xi^k = \xi \cdots \xi$ is a product in the symmetric algebra, while on the right hand side it is taken in the enveloping algebra. (Note that the elements ξ^k with $\xi \in \mathfrak{g}$ span $S^k(\mathfrak{g})$, by polarization.) The symmetrization map is the direct analogue of the quantization map $q\colon \wedge(V) \to \mathrm{Cl}(V)$ for Clifford algebras, which was given by symmetrization in the graded sense.

THEOREM 2.4 (Poincaré-Birkhoff-Witt, version III). *The symmetrization map* $\mathrm{sym}\colon S(\mathfrak{g}) \to U(\mathfrak{g})$, *is an isomorphism of filtered vector spaces.*

REMARK 2.5. It it rather easy to see that $\mathrm{sym}\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ is surjective: For this, it suffices to show that the map $S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ is surjective. But this follows e.g. since the elements $e_{i_1} \cdots e_{i_k}$ for weakly increasing sequences $i_1 \le \cdots i_k$ span $U(\mathfrak{g})$. Hence, the difficult part of the Poincaré-Birkhoff-Witt theorem is to show that the map sym is injective.

THEOREM 2.6 (Poincaré-Birkhoff-Witt, version IV). *There exists a* $\mathfrak{g}$-*representation* $\varrho\colon \mathfrak{g} \to \mathrm{End}(S(\mathfrak{g}))$, *with the property*

$$\varrho(\zeta)(\zeta^n) = \zeta^{n+1} \tag{62}$$

for all $\zeta \in \mathfrak{g},\ n \geq 0$.

To the equivalence with version III, suppose first that $\mathrm{sym}\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ is known to be an isomorphism. Let ϱ^L be the left regular representation of $\mathfrak{g}$ on $U(\mathfrak{g})$, $\varrho^L(\xi)x = \xi x$, and let ϱ be the $\mathfrak{g}$-representation on $S(\mathfrak{g})$ corresponding to ϱ^L under the isomorphism sym. Equation (62) follows from $\varrho^L(\zeta)\,\mathrm{sym}(\zeta^n) = \zeta\,\zeta^n = \zeta^{n+1} = \mathrm{sym}(\zeta^{n+1})$. Conversely, given a $\mathfrak{g}$-representation ϱ on $S(\mathfrak{g})$ satisfying (62), extend to an algebra morphism $\varrho\colon U(\mathfrak{g}) \to \mathrm{End}(S(\mathfrak{g}))$ and define a *symbol map*

$$\sigma\colon U(\mathfrak{g}) \to S(\mathfrak{g}),\ x \mapsto \varrho(x).1.$$

Then

$$\sigma(\mathrm{sym}(\zeta^n)) = \varrho(\zeta^n).1 = \varrho(\zeta)^n.1 = \zeta^n,$$

proving $\sigma \circ \mathrm{sym} = \mathrm{id}_{S(\mathfrak{g})}$. Since sym is surjective, it follows that sym is an isomorphism.

A beautiful direct proof of version IV of the Poincaré-Birkhoff-Witt theorem was obtained Emanuela Petracci [**56**] in 2003. We will present this proof in Section §5.5 below. In fact, Petracci's argument yields the following explicit formula for the representation:

$$\varrho(\zeta)(\xi^n) = \sum_{k=0}^{n} \binom{n}{k} B_k\, \xi^{n-k}\, \mathrm{ad}_\xi^k(\zeta), \tag{63}$$

where B_n are the Bernoulli numbers, defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

Notice that if $\xi = \zeta$, only the term $k = 0$ contributes to (63) and we get (62) as required. The verification that (63) defines a Lie algebra representation on $S(\mathfrak{g})$ is the main task in this approach, and will be carried out in §5.5).

Recall that the Bernoulli numbers for odd $n \geq 3$ are all zero, while $B_0 = 1,\ B_1 = -\frac{1}{2}$, and

$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66}, \ldots.$$

One deduces the following expressions in low degrees:

$$\begin{aligned}
\varrho(\zeta)(1) &= \zeta,\\
\varrho(\zeta)(\xi) &= \xi\zeta - \tfrac{1}{2}[\xi,\zeta],\\
\varrho(\zeta)(\xi^2) &= \xi^2\zeta - \xi[\xi,\zeta] + \tfrac{1}{6}[\xi,[\xi,\zeta]]\\
&\cdots
\end{aligned}$$

Once the Poincaré-Birkhoff-Witt theorem is in place, we may use the symmetrization map sym: $S(\mathfrak{g}) \to U(\mathfrak{g})$ to transfer the non-commutative product on $U(\mathfrak{g})$ to a product $*$ on $S(\mathfrak{g})$. By definition of symmetrization, and of the enveloping algebra, we have

$$\xi_1\xi_2 = \tfrac{1}{2}(\xi_1 * \xi_2 + \xi_2 * \xi_1), \quad [\xi_1, \xi_2] = \xi_1 * \xi_2 - \xi_2 * \xi_1.$$

Hence

$$\xi_1 * \xi_2 = \xi_1\xi_2 + \tfrac{1}{2}[\xi_1, \xi_2].$$

The triple product is already much more complicated. One finds, after cumbersome computation,

$$\begin{aligned}\xi_1 * \xi_2 * \xi_3 = \xi_1\xi_2\xi_3 &+ \frac{\xi_3[\xi_1,\xi_2] + \xi_1[\xi_2,\xi_3] + \xi_2[\xi_1,\xi_3]}{2} \\ &+ \frac{[\xi_1,[\xi_2,\xi_3]] - [\xi_3,[\xi_1,\xi_2]]}{6}.\end{aligned}$$

REMARK 2.7. Similar to the discussion for Clifford algebras, the isomorphism $\mathrm{gr}(U(\mathfrak{g})) \cong S(\mathfrak{g})$ induces the structure of graded Poisson algebra on $S(\mathfrak{g})$. The Poisson structure is determined by $\{\xi_1, \xi_2\} = [\xi_1, \xi_2]$ for generators $\xi_1, \xi_2 \in \mathfrak{g}$, hence it coincides with the Kirillov-Poisson structure on the space of polynomial functions on $\mathfrak{g}^*$, see Example 3.3 in §2.3.

3. $U(\mathfrak{g})$ as left-invariant differential operators

For any manifold M, let $\mathfrak{D}(M)$ denote the algebra of differential operators on M (cf. Section §2.3.1). Given an action of a Lie group G on M, one can consider the subalgebra $\mathfrak{D}(M)^G$ of differential operators which commute with the G-action. The isomorphism (defined by the principal symbol)

$$\sigma^\bullet \colon \mathrm{gr}^\bullet \mathfrak{D}(M) \to \Gamma^\infty(M, S^\bullet(TM))$$

is G-equivariant, and restricts to an isomorphism on G-invariants,

$$(\mathrm{gr}^\bullet \mathfrak{D}(M))^G \to \Gamma^\infty(M, S^\bullet(TM))^G.$$

We also have an injection $\mathrm{gr}^\bullet(\mathfrak{D}(M)^G) \to (\mathrm{gr}^\bullet \mathfrak{D}(M))^G$, but for non-compact Lie groups and ill-behaved actions this need not be an isomorphism.

Consider now the special case of the left-action of G on itself. Let $\mathfrak{D}^L(G)$ denote the differential operators on G which commute with left translation. The Lie algebra isomorphism

$$\mathfrak{g} \mapsto \mathfrak{X}^L(G), \ \xi \mapsto \xi^L,$$

where $\mathfrak{g}$ is teh Lie algebra of G, extends to an algebra homomorphism $T(\mathfrak{g}) \to \mathfrak{D}^L(G)$, which vanishes on the ideal $\mathcal{I}$. Hence we get an induced algebra homomorphism

$$U(\mathfrak{g}) \to \mathfrak{D}^L(G),$$

taking the image of $\xi \in \mathfrak{g} = T^1(\mathfrak{g})$ to ξ^L. The Poincaré-Birkhoff-Witt theorem now has a differential-geometric interpretation.

THEOREM 3.1 (Poincaré-Birkhoff-Witt, version V). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathfrak{g}$, and G a Lie group integrating $\mathfrak{g}$. The canonical map $U(\mathfrak{g}) \to \mathfrak{D}^L(G)$ is an isomorphism of algebras.*

PROOF. We have

$$\Gamma^\infty(G, S^\bullet(TG))^L = S(T_eG) = S(\mathfrak{g}),$$

where the superscript G indicates invariants under the action by left multiplication. Consider the composition of maps

$$S^k\mathfrak{g} \to U^{(k)}\mathfrak{g} \to \mathfrak{D}^{(k),L}(G) \xrightarrow{\sigma} S^k\mathfrak{g},$$

where the first map is symmetrization. As noted earlier, the associated graded map $S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ is independent of this choice, and we obtain a sequence of morphisms of graded algebras,

$$S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g})) \to \mathrm{gr}(\mathfrak{D}^L(G)) \to S(\mathfrak{g}). \tag{64}$$

The composition of these maps is the identity on $S(\mathfrak{g})$, since it is clearly the identity on $\mathfrak{g}$. The last map is injective (since it comes from the inclusion of $\mathrm{gr}(\mathfrak{D}^L(G))$ into $\mathrm{gr}(\mathfrak{D}(G))^L \cong S(\mathfrak{g}))$), hence it must be an isomorphism. The first map is surjective (since the e_I span $U(\mathfrak{g})$), hence it too must be an isomorphism. But then the middle map must be an isomorphism as well. $\square$

Note that the proof also gives Version III, and hence als versions I,II,IV, of the Poincaré-Birkhoff-Witt theorem for the case of finite-dimensional real Lie algebras. Recall however that these argument depend on Lie's third theorem, which is not considered an elementary result. By contrast, the argument in §5.5 of the Poincaré-Birkhoff-Witt theorem is purely algebraic.

4. The enveloping algebra as a Hopf algebra

4.1. Hopf algebras. An algebra may be viewed as a triple $(\mathcal{A}, m, i)$ consisting of a vector space $\mathcal{A}$, together with linear maps $m\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ (the *multiplication*) and $i\colon \mathbb{K} \to \mathcal{A}$ (the *unit*), such that

$$\begin{aligned} m\circ(m\otimes 1) &= m\circ(1\otimes m) \quad \text{(Associativity)},\\ m\circ(i\otimes 1) &= m\circ(1\otimes i) = 1 \quad \text{(Unit property)}. \end{aligned} \tag{65}$$

It is called *commutative* if $m\circ\mathcal{T} = m$, where $\mathcal{T}\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{A},\ x\otimes x' \mapsto x'\otimes x$ exchanges the two factors. A coalgebra is defined similar to an algebra, but with 'arrows reversed':

DEFINITION 4.1. A *coalgebra* is a vector space $\mathcal{A}$, together with linear maps

$$\Delta\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{A},\quad \epsilon\colon \mathcal{A}\to\mathbb{K}$$

called *comultiplication* and *counit*, such that

$$(\Delta\otimes 1)\circ\Delta = (1\otimes\Delta)\circ\Delta \quad \text{(Coassociativity)},$$

$$(\epsilon\otimes 1)\circ\Delta = (1\otimes\epsilon)\circ\Delta = 1 \quad \text{(Counit property)}.$$

It is called *cocommutative* if $\mathcal{T}\circ\Delta = \Delta$.

It is fairly obvious from the definition that the dual of any coalgebra is an algebra. By contrast, the dual of an algebra $\mathcal{A}$ is not a coalgebra, in general, since the dual map $m^*\colon \mathcal{A}^* \to (\mathcal{A}\otimes\mathcal{A})^*$ does not take values in $\mathcal{A}^*\otimes\mathcal{A}^*$ unless $\dim\mathcal{A} < \infty$. There is an obvious notion of morphism of coalgebras; for example the counit provides such a morphism.

A Hopf algebra is a vector space with compatible algebra and coalgebra structures, as follows:

Definition 4.2. A *Hopf algebra* is a vector space $\mathcal{A}$, together with maps $m\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ (multiplication), $i\colon \mathbb{K}\to\mathcal{A}$ (unit), $\Delta\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ (comultiplication), $\epsilon\colon \mathcal{A}\to\mathbb{K}$ (counit), $\mathsf{s}\colon \mathcal{A}\to A$ (antipode), such that

(1) $(\mathcal{A}, m, i)$ is an algebra,
(2) $(\mathcal{A}, \Delta, \epsilon)$ is a coalgebra,
(3) Δ and ϵ are algebra morphisms,
(4) s is a linear isomorphism and has the property,

$$m\circ(1\otimes\mathsf{s})\circ\Delta = m\circ(\mathsf{s}\otimes 1)\circ\Delta = i\circ\epsilon.$$

Remarks 4.3. The condition (3) that Δ, ϵ are algebra morphisms are equivalent to m, i being coalgebra morphisms. Indeed, both properties are expressed by the formulas

$$\begin{aligned}
\Delta\circ m &= (m\otimes m)\circ(1\otimes\mathcal{T}\otimes 1)(\Delta\otimes\Delta),\\
\epsilon\otimes\epsilon &= \epsilon\circ m,\\
\Delta\circ i &= i\otimes i,\\
\epsilon\circ i &= 1.
\end{aligned}$$

Furthermore, it is automatic [**41**, Theorem III.3.4] that s is an algebra anti-homomorphism as well as a coalgebra anti-homomorphism, and that $\mathsf{s}\circ i = i,\ \ \epsilon\circ\mathsf{s} = \epsilon$.

Hopf algebras may be viewed as algebraic counterparts, or rather generalizations, of groups. Indeed any group defines a Hopf algebra:

Example 4.4. Let Γ be any group, and $\mathbb{K}[\Gamma]$ its *group algebra*. Thus $\mathbb{K}[\Gamma]$ has vector space basis Γ, with unit $i(1) = e$ and with the multiplication

$$m(g\otimes g') = gg',\quad i(1) = e,$$

extended to general elements $\sum_{g\in\Gamma} a_g\, g \in \mathbb{K}[\Gamma]$ by linearity. The algebra structure extends to a Hopf algebra structure by putting

$$\Delta(g) = g\otimes g,\quad \epsilon(\sum_{g\in\Gamma} a_g\, g) = a_e,\quad \mathsf{s}(g) = g^{-1}.$$

In this example, the group Γ can be recovered from the Hopf algebra $\mathbb{K}[\Gamma]$ as the elements satisfying $\Delta(x) = x\otimes x$. This motivates the following notion.

Definition 4.5. An element x of a Hopf algebra $(\mathcal{A}, m, i, \Delta, \epsilon, s)$ is called *group-like* if $\Delta(x) = x\otimes x$.

PROPOSITION 4.6. *The set of group-like elements is a group, with multiplication m, inverse $x^{-1} = \mathsf{s}(x)$, and group unit $e = i(1)$.*

PROOF. Since Δ is an algebra morphism, the product of two group-like elements is again group-like. Since s is a coalgebra anti-homomorphism one has, for any group-like element x,

$$\Delta(\mathsf{s}(x)) = (\mathsf{s} \otimes \mathsf{s})(\mathcal{T}(\Delta(x))) = (\mathsf{s} \otimes \mathsf{s})(\Delta(x)) = \mathsf{s}(x) \otimes \mathsf{s}(x),$$

so that $\mathsf{s}(x)$ is group-like. By applying $\epsilon \otimes 1$ to the definition, one verifies that group-like elements satisfy $\epsilon(x) = 1$. This then shows

$$m(x \otimes s(x)) = m(1 \otimes \mathsf{s})(\Delta(x)) = i(\epsilon(x)) = i(1) = e,$$

and similarly $m(\mathsf{s}(x) \otimes x) = e$. Hence $\mathsf{s}(x) = x^{-1}$. $\square$

EXAMPLE 4.7 (Finite groups). Let $\mathcal{A} = C(\Gamma, \mathbb{K})$ be the algebra of functions on a finite group Γ, with m the pointwise multiplication and i given is the inclsuon as constant functions. Define a comultiplication

$$\Delta \colon C(\Gamma, \mathbb{K}) \to C(\Gamma, \mathbb{K}) \otimes C(\Gamma, \mathbb{K}) = C(\Gamma \times \Gamma, \mathbb{K}),$$

a counit ϵ, and an antipode s by

$$\Delta(f)(g_1, g_2) = f(g_1 g_2), \quad \epsilon(f) = f(e), \quad \mathsf{s}(f)(g) = f(g^{-1}).$$

Then $(\mathcal{A}, m, i, \Delta, \epsilon, \mathsf{s})$ is a finite-dimensional Hopf algebra. Its group-like elements are given by $\widehat{\Gamma} = \mathrm{Hom}(\Gamma, \mathbb{K}^\times) \subset \mathcal{A}$.

REMARK 4.8. One can show [**41**, Proposition III.3.3] that the dual of any *finite-dimensional* Hopf algebra $(\mathcal{A}, m, i, \Delta, \epsilon, \mathsf{s})$ is a Hopf-algebra

$$(\mathcal{A}^*, \Delta^*, \epsilon^*, m^*, i^*, \mathsf{s}^*).$$

For instance, if Γ is a finite group the Hopf algebras $\mathbb{K}[\Gamma]$ and $C(\Gamma, \mathbb{K})$ are dual.

REMARK 4.9. Any Hopf algebra $\mathcal{A}$ gives rise to a group

$$\Gamma_{\mathcal{A}} = \mathrm{Hom}_{\mathrm{alg}}(\mathcal{A}, \mathbb{K})$$

(algebra homomorphisms) with product

$$\phi_1 \phi_2 = (\phi_1 \otimes \phi_2) \circ \Delta,$$

inverse $\phi^{-1} = \phi \circ s$, and group unit $e = \epsilon$. If $\mathcal{A} = C(\Gamma, \mathbb{K})$ for a finite group Γ, then the map

$$\Gamma \to \Gamma_{\mathcal{A}}, \ g \mapsto \left[\mathrm{ev}_g \colon f \mapsto f(g) \right]$$

is an isomorphism. (Tannaka-Krein duality, see e.g. [**15**, Chapter III.7].)

4.2. Hopf algebra structure on $S(E)$. Let E be a vector space (possible $\dim E = \infty$), and $(S(E), m, i)$ the symmetric algebra over E. Any morphism of vector spaces induces an algebra morphism of their symmetric algebras; in particular the diagonal inclusion $E \to E \oplus E$ defines an algebra morphism

$$\Delta\colon S(E) \to S(E) \otimes S(E) = S(E \oplus E).$$

Let $\mathsf{s}\colon S(E) \to S(E)$ the canonical anti-automorphism (equal to $v \mapsto -v$ on $E \subset S(E)$), and let $\epsilon\colon S(E) \to \mathbb{K}$ be the augmentation map. Then $(S(E), m, i, \Delta, \epsilon, \mathsf{s})$ is a Hopf algebra. Since $\Delta(v) = v \otimes 1 + 1 \otimes v$, we have

$$\Delta(v^k) = (v \otimes 1 + 1 \otimes v)^k = \sum_{j=0}^{k} \binom{k}{j} v^{k-j} \otimes v^j.$$

By polarization, elements of the form v^k span all of $S^k(E)$, hence these formulas determine Δ.

Since all the structure maps preserve gradings, they extend to the degree completion

$$\overline{S}(E) = \prod_{k=0}^{\infty} S^k(E)$$

given by the direct product. Thus $\overline{S}(E)$ is again a Hopf algebra. For any $v \in V$, the exponential

$$e^v = \sum_{k=0}^{\infty} \frac{1}{k!} v^k$$

is a group-like element of $\overline{S}(E)$. Indeed

$$\Delta(e^v) = e^v \otimes e^v$$

by the formulas for $\Delta(v^k)$. For the multiplication map, counit and antipode we similarly have

$$m(e^v \otimes e^{v'}) = e^v e^{v'}, \quad \epsilon(e^v) = 1, \quad \mathsf{s}(e^v) = e^{-v}.$$

4.3. Hopf algebra structure on $U(\mathfrak{g})$. Let $\mathfrak{g}$ be a Lie algebra, and $(U(\mathfrak{g}), m, i)$ its enveloping algebra $U(\mathfrak{g})$. The diagonal inclusion $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$, $\xi \mapsto \xi \oplus \xi$ is a Lie algebra morphism, hence it extends to an algebra morphism

$$\Delta\colon U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}) = U(\mathfrak{g} \oplus \mathfrak{g})$$

Together with the counit $\epsilon\colon U(\mathfrak{g}) \to \mathbb{K}$ given as the augmentation map, and the antipode $\mathsf{s}\colon U(\mathfrak{g}) \to U(\mathfrak{g})$ given by the canonical anti-automorphism of $U(\mathfrak{g})$, we find:

THEOREM 4.10. *$(U(\mathfrak{g}), m, i, \Delta, \epsilon, \mathsf{s})$ is a cocommutative Hopf algebra.*

The example $S(E)$ from the last section is a special case, thinking of E as a Lie algebra with zero bracket.

PROOF. By definition, Δ and ϵ are algebra homomorphisms. The coassociativity of Δ follows because both $(\Delta \otimes 1) \circ \Delta$ and $(1 \otimes \Delta) \circ \Delta$ are the maps

$$U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g}) = U(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g})$$

induced by the triagonal inclusion. The counital properties of ϵ are equally clear. It remains to check the properties of the antipode. The property $m \circ (1 \otimes \mathsf{s}) \circ \Delta = i \circ \epsilon$ is clearly true on scalars $i(\mathbb{K}) \subset U(\mathfrak{g})$, while the general case follows by induction on the filtration degree: if the property holds on $x \in U^{(k)}(\mathfrak{g})$, and if $\xi \in \mathfrak{g}$, then $i(\epsilon(\xi x)) = 0$ since ξx is in the augmentation ideal, and also

$$\begin{aligned} m((1 \otimes \mathsf{s})(\Delta(\xi x))) &= m((1 \otimes \mathsf{s})((\xi \otimes 1 + 1 \otimes \xi)\Delta(x)) \\ &= m((\xi \otimes 1)(1 \otimes \mathsf{s})(\Delta(x)) - m((1 \otimes \mathsf{s})(\Delta(x))(1 \otimes \xi)) \\ &= \xi m((1 \otimes \mathsf{s})(\Delta(x)) - m((1 \otimes \mathsf{s})(\Delta(x)))\xi \\ &= \xi i(\epsilon(x)) - i(\epsilon(x))\xi \\ &= 0 \end{aligned}$$

where we used that s is an anti-homomorphism. The property $m \circ (\mathsf{s} \otimes 1) \circ \Delta = i \circ \epsilon$ is verified similarly. Cocommutativity $\mathcal{T} \circ \Delta = \Delta$ follows since it holds on generators $v \in \mathfrak{g}$, and since $\mathcal{T} \colon U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is an algebra homomorphism (induced by the Lie algebra isomorphism $(\xi_1, \xi_2) \mapsto (\xi_2, \xi_1)$). □

To summarize, we can think of $U(\mathfrak{g})$ as an algebraic analogue or substitute for the Lie group G integrating $\mathfrak{g}$. (The cocommutativity of $U(\mathfrak{g})$ is parallel to the fact that $C(G, \mathbb{K})$ is a commutative algebra.) This point of view is taken in the definition of quantum groups, which are not actually groups but are defined as suitable Hopf algebras.

It is obvious that for $\mathfrak{g}$ non-abelian, the symmetrization map $\mathrm{sym} \colon S(\mathfrak{g}) \to U(\mathfrak{g})$ does not intertwine the multiplications m. On the other hand, it intertwines all the other Hopf algebra structure maps:

PROPOSITION 4.11. *The symmetrization map* $\mathrm{sym} \colon S(\mathfrak{g}) \to U(\mathfrak{g})$ *intertwines the comultiplications* Δ*, counits* ϵ*, antipodes* s*, and units* i *of the Hopf algebras* $S(\mathfrak{g})$ *and* $U(\mathfrak{g})$*. In particular,* sym *is a coalgebra homomorphism.*

PROOF. It is clear that $\mathrm{sym} \circ i = i$. The symmetrization map is functorial with respect to Lie algebra homomorphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$. Functoriality for the diagonal inclusion $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ shows that sym intertwines Δ, while functoriality relative to the projection $\mathfrak{g} \mapsto \{0\}$ implies that sym intertwines ϵ. Finally, let $\xi_1, \ldots, \xi_k \in \mathfrak{g}$, and $\xi_1 \cdots \xi_k \in S(\mathfrak{g})$ its product in the enveloping algebra. Then

$$\begin{aligned}\operatorname{sym}(\mathsf{s}(\xi_1,\ldots,\xi_k)) &= (-1)^k \operatorname{sym}(\xi_1,\ldots,\xi_k)\\ &= \frac{(-1)^k}{k!}\sum_{s\in\mathfrak{S}_k}\xi_{s(1)}\cdots\xi_{s(k)}\\ &= \frac{1}{k!}\mathsf{s}\Big(\sum_{s\in\mathfrak{S}_k}\xi_{s(k)}\cdots\xi_{s(1)}\Big)\\ &= \mathsf{s}(\operatorname{sym}(\xi_1,\ldots,\xi_k)).\end{aligned}$$

□

REMARK 4.12. There is a formula for $\Delta(\xi^k)$, $\xi \in \mathfrak{g}$, similar to that for the symmetric algebra, see §5.4.2. Since the structure maps of the Hopf algebra are filtration preserving, they pass to the colimit $\overline{U}(\mathfrak{g}) = \lim_{k\to\infty} U^{(k)}(\mathfrak{g})$. As for the symmetric algebra, elements $e^\xi = \sum_{k=0}^\infty \frac{1}{k!}\xi^k$, $\xi \in \mathfrak{g}$ are well-defined group-like elements of $\overline{U}(\mathfrak{g})$.

4.4. Primitive elements. It is in fact possible to recover $\mathfrak{g}$ from $U(\mathfrak{g})$. For this we need the following

DEFINITION 4.13. An element x of a Hopf algebra $(\mathcal{A}, m, i, \Delta, \epsilon, \mathsf{s})$ is called *primitive* if $\Delta(x) = x\otimes 1 + 1\otimes x$. Let $P(\mathcal{A})$ denote the space of primitive elements.

LEMMA 4.14. *For any Hopf algebra $\mathcal{A}$, the space of primitives $P(\mathcal{A})$ is a Lie subalgebra under commutator.*

PROOF. Suppose x, y are primitive. Since Δ is an algebra homomorphism,

$$\begin{aligned}\Delta(xy-yx) &= \Delta(x)\Delta(y) - \Delta(y)\Delta(x)\\ &= (x\otimes 1 + 1\otimes x)(y\otimes 1 + 1\otimes y) - (y\otimes 1 + 1\otimes y)(x\otimes 1 + 1\otimes x)\\ &= (xy-yx)\otimes 1 + 1\otimes(xy-yx),\end{aligned}$$

which shows that $xy - yx$ is primitive. □

For any vector space E, we have $E \subset P(S(E))$ by definition of the coproduct. More generally, for any Lie algebra we have $\mathfrak{g}\subset P(U(\mathfrak{g}))$.

LEMMA 4.15. *For any vector space E over $\mathbb{K}$, the set of primitive elements in the symmetric algebra is $P(S(E)) = E$.*

PROOF. It is clear that $P(S(E))$ is a graded subspace of $S(E)$ containing E. Since elements of degree 0 cannot be primitive, it remains to show that there are no primitive elements of degree $k > 1$. Given $\mu \in E^*$, let

$$(\mathrm{id}_{S(E)}\otimes\mu)\colon S(E)\otimes S(E) \to S(E)\otimes\mathbb{K} = S(E)$$

be the identity map on the first factor and the pairing with $\mu \in E^* \subset S(E)^*$ on the second factor. Since this map vanishes on $S(E)\otimes S^k(E)$ for $k \neq 0$, we have

$$(\mathrm{id}_{S(E)} \otimes \mu)\Delta(v^n) = (\cdot \otimes \mu) \sum_{k=0}^{n} \binom{n}{k} v^{n-k} \otimes v^k$$
$$= n v^{n-1} \langle \mu, v \rangle.$$

We hence see that $(\cdot \otimes \mu) \circ \Delta = i_S(\mu)$. If $x \in S^k(E)$ is primitive, it follows that

$$\iota_S(\mu)x = (\mathrm{id}_{S(E)} \otimes \mu)(1 \otimes x + x \otimes 1) = \langle \mu, x \rangle,$$

If $k > 1$, the right hand side vanishes. This shows $x = 0$. □

According to the Poincaré-Birkhoff-Witt theorem, the symmetrization map is an isomorphism of coalgebras $S(\mathfrak{g}) \to U(\mathfrak{g})$. Since the definition of primitive elements only involves the comultiplication, we may conclude that

$$P(U(\mathfrak{g})) = \mathfrak{g},$$

an isomorphism of Lie algebras. In summary, $\mathfrak{g}$ can be recovered from the Hopf algebra structure of $U(\mathfrak{g})$.

4.5. Coderivations. A derivation of an algebra $(\mathcal{A}, m, i)$ is a linear map $D\colon \mathcal{A} \to \mathcal{A}$ satisfying $D \circ m = m \circ (D \otimes 1 + 1 \otimes D)$. Similarly one defines:

Definition 4.16. A *coderivation* of a coalgebra $(\mathcal{A}, \Delta, \epsilon)$ is a linear map $C\colon \mathcal{A} \to \mathcal{A}$ satisfying

$$\Delta \circ C = (C \otimes 1 + 1 \otimes C) \circ \Delta.$$

The space of derivations of an algebra is a Lie algebra under commutator. The same is true for coderivations of a coalgebra:

Lemma 4.17. *The space of coderivations of a coalgebra is a Lie algebra under commutator.*

Proof. If C_1, C_2 are two coderivations, then

$$\Delta \circ C_1 \circ C_2 = (C_1 \otimes 1 + 1 \otimes C_1) \circ (C_2 \otimes 1 + 1 \otimes C_2) \circ \Delta$$
$$= (C_1 \circ C_2 \otimes 1 + 1 \otimes C_1 \circ C_2 + C_1 \otimes C_2 + C_2 \otimes C_1) \circ \Delta$$

Subtracting a similar equation for $\Delta \circ C_2 \circ C_1$, one obtains the derivation property of $[C_1, C_2] = C_1 \circ C_2 - C_2 \circ C_1$. □

Proposition 4.18. *For any Hopf algebra $(\mathcal{A}, m, i, \Delta, \epsilon)$, the map*

$$P(\mathcal{A}) \times \mathcal{A} \to \mathcal{A}$$

given by restriction of m is an action of the Lie algebra of primitive elements by coderivations.

Proof. For $\xi \in P(\mathcal{A})$ and $x \in \mathcal{A}$,

$$\Delta(\xi x) = \Delta(\xi)\Delta(x) = (\xi \otimes 1 + 1 \otimes \xi)\Delta(x).$$ □

Corollary 4.19. *The left regular representation of* $\mathfrak{g}$ *on* $U(\mathfrak{g})$

$$\varrho^L \colon \mathfrak{g} \to \mathrm{End}(U(\mathfrak{g})),$$

given by $\varrho^L(\xi)x = \xi x$ *is an action by coderivations of* $U(\mathfrak{g})$.

Proof. Immediate from the Proposition, since the elements of $\mathfrak{g} \subset U(\mathfrak{g})$ are primitive. $\square$

4.6. Coderivations of $S(E)$. Let E be a vector space. Recall that the space of derivations of $S(E)$ is isomorphic to the space of linear maps $E \to S(E)$, since any such map extends uniquely as a derivation. Thus

$$\mathrm{Der}(S(E)) \cong \mathrm{Hom}(E, S(E))$$

as graded vector spaces. Dually, one expects that the space of coderivations of the co-algebra $S(E)$ is isomorphic to the space $\mathrm{Hom}(S(E), E)$. In more geometric terms, we may think of the elements of

$$S(E)^* = \mathrm{Hom}(S(E), \mathbb{K}) = \prod_{k=0}^{\infty} S^k(E)^*$$

as *formal functions* on E, i.e. Taylor expansions at 0 of smooth functions on E. Accordingly we think of elements $X \in \mathrm{Hom}(S(E), E)$ as *formal vector fields*. Any formal vector field is determined by its action on elements $v^n \in S^n(E)$, for $n = 0, 1, \ldots$. It is convenient to introduce the 'generating function' $e^{tv} = \sum_{k=0}^{\infty} \frac{t^k}{k!} v^k \in S(E)[[t]]$; then X is determined by $X(e^{tv}) \in E[[t]]$. A coderivation C of $S(E)$ defines a derivation C^* of the algebra $S(E)^*$, and hence should correspond to a formal vector field.

theorem 4.20. *There is a canonical isomorphism*

$$\mathrm{Coder}(S(E)) \cong \mathrm{Hom}(S(E), E)$$

between the space of coderivations of $S(E)$ *and the space of formal vector fields on* E*. The isomorphism takes* $X \in \mathrm{Hom}(S(E), E)$ *to the coderivation*

$$C = m \circ (1 \otimes X) \circ \Delta.$$

Proof. It is convenient to work with the generating function e^{tv}. Since $\Delta(e^{tv}) = e^{tv} \otimes e^{tv} \in S(V \oplus V)[[t]]$, the formula relating X and C reads,

$$C(e^{tv}) = e^{tv} X(e^{tv}).$$

Explicitly, we have

$$C(v^n) = \sum_{k=0}^{n} \binom{n}{k} v^k X(v^{n-k})$$

for all $n = 0, 1, 2, \ldots$. We first show that if C is a coderivation, then $X(e^{tv}) := e^{-tv} C(e^{tv})$ lies in $E[[t]] \subset S(E)[[t]]$. Equivalently, we show that

$X(e^{tv})$ is primitive:

$$\begin{aligned}\Delta(X(e^{tv})) &= \Delta(e^{-tv})\Delta(C(e^{tv}))\\ &= (e^{-tv}\otimes e^{-tv})\cdot(C\otimes 1+1\otimes C)\Delta(e^{tv})\\ &= (e^{-tv}\otimes e^{-tv})\cdot(C(e^{tv})\otimes e^{tv}+e^{tv}\otimes C(e^{tv}))\\ &= e^{-tv}C(e^{tv})\otimes 1+1\otimes e^{-tv}C(e^{tv})\\ &= X(e^{tv})\otimes 1+1\otimes X(e^{tv}).\end{aligned}$$

Conversely, if $X\in \mathrm{Hom}(S(E),E)$, a similar calculation shows that $C(e^{tv}):=e^{tv}X(e^{tv})$ defines a coderivation:

$$\begin{aligned}\Delta(C(e^{tv})) &= (e^{tv}\otimes e^{tv})\Delta(X(e^{tv}))\\ &= (e^{tv}\otimes e^{tv})(X(e^{tv})\otimes 1+1\otimes X(e^{tv}))\\ &= C(e^{tv})\otimes e^{tv}+e^{tv}\otimes C(e^{tv})\\ &= (C\otimes 1+1\otimes C)\circ\Delta(e^{tv}).\end{aligned}$$

□

PROPOSITION 4.21. *The Lie bracket on* $\mathrm{Hom}(S(E),E)$ *induced by the isomorphism with the Lie algebra* $\mathrm{Coder}(S(E))$ *of coderivations reads,*

$$[X_1,X_2](e^{tv}) = X_1(e^{tv}X_2(e^{tv})) - X_2(e^{tv}X_1(e^{tv})). \tag{66}$$

PROOF. For any $Y\in E$ we have

$$\begin{aligned}C_1(e^{tv}Y) &= \frac{\partial}{\partial s}|_{s=0}C_1(e^{tv+sY})\\ &= \frac{\partial}{\partial s}|_{s=0}e^{tv+sY}X_1(e^{tv+sY})\\ &= e^{tv}X_1(e^{tv})Y+e^{tv}X_1(e^{tv}Y).\end{aligned}$$

Putting $Y=X_2(e^{tv})\in E[[t]]$ we find

$$C_1(C_2(e^{tv})) = e^{tv}X_1(e^{tv})X_2(e^{tv}) + e^{tv}X_1(e^{tv}X_2(e^{tv}))$$

and consequently

$$[C_1,C_2](e^{tv}) = e^{tv}X_1(e^{tv}X_2(e^{tv})) - e^{tv}X_2(e^{tv}X_2(e^{tv})).$$ □

5. Petracci's proof of the Poincaré-Birkhoff-Witt theorem

Our goal in this Section is to prove version IV of the Poincaré-Birkhoff-Witt theorem. We will indeed prove a more precise version, Petracci's theorem 5.1 below, explicitly describing the $\mathfrak{g}$-representation on $S(\mathfrak{g})$ corresponding to the left regular representation on $U(\mathfrak{g})$.

To simplify notation, we will omit the parameter t from the 'generating functions' e^{tv}, and simply write e.g.

$$C(e^v) = e^vX(e^v).$$

This is a well-defined equality in $\overline{S}(E)$ if X (hence C) is homogeneous of some fixed degree; otherwise we view this identity as an equality of formal power series in t, obtained by replacing v with tv.

5.1. A $\mathfrak{g}$-representation by coderivations. The main idea in Petracci's approach to the Poincaré-Birkhoff-Witt theorem is to define a $\mathfrak{g}$-representation on $S(\mathfrak{g})$, which under symmetrization $\mathrm{sym}\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ goes to the left-regular representation

$$\varrho^L\colon \mathfrak{g} \to \mathrm{End}(U(\mathfrak{g})), \quad \varrho^L(\zeta).x = \zeta x$$

of $\mathfrak{g}$ on $U(\mathfrak{g})$. The representation should be by coderivations of $S(\mathfrak{g})$, since $\varrho^L(\zeta)$ is a coderivation (cf. Corollary 4.19) and sym is a coalgebra homomorphism (cf. Proposition 4.11). Equivalently, we are interested in Lie algebra homomorphisms

$$\varrho\colon \mathfrak{g} \to \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g}), \quad \zeta \mapsto X^\zeta.$$

Using differential geometry, one can make a guess for X^ζ, as follows. Suppose $\mathbb{K} = \mathbb{R}$, and let G be a Lie group integrating $\mathfrak{g}$. The left regular representation of $\mathfrak{g}$ on $U(\mathfrak{g})$ can be thought of as a counterpart to the left-action of G on itself, and the symmetrization map is an algebraic counterpart[1] to the exponential map $\exp\colon \mathfrak{g} \to G$. The exponential map is a local diffeomorphism on an open dense subset $\mathfrak{g}_\natural \subset \mathfrak{g}$. We may view the pull-back $\exp^*(\zeta^R) \in \mathfrak{X}(\mathfrak{g}_\natural)$ as a $\mathfrak{g}$-valued function on $\mathfrak{g}_\natural$. Explicit calculation (see Section 4 in Appendix C) gives that this function is

$$\xi \mapsto \phi(\mathrm{ad}_\xi)\zeta,$$

with $\phi(z) = \frac{z}{e^z - 1}$. This suggests defining $X^\zeta \in \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g})$ by $X^\zeta(e^\xi) = \phi(\mathrm{ad}_\xi)\zeta$, and this formula makes sense for any field $\mathbb{K}$. Petracci's theorem below shows that this is indeed the correct choice.

THEOREM 5.1 (Petracci). *Let $X^\zeta \in \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g})$, $\zeta \in \mathfrak{g}$ be the formal vector fields defined by*

$$X^\zeta(e^\xi) = \phi(\mathrm{ad}_\xi)\zeta$$

where $\phi(z) = \frac{z}{e^z - 1}$. Then

$$[X^{\zeta_1}, X^{\zeta_2}] = X^{[\zeta_1, \zeta_2]},$$

and ϕ is the unique formal power series with $\phi(0) = 1$ having this property. Hence, the formal vector fields X^ζ define a representation by coderivations.

The proof will be given in 5.3, after some preparations.

[1] More concretely, the symmetric algebra $S(\mathfrak{g})$ may be identified with the convolution algebra of distributions (generalized measures) on $\mathfrak{g}$ supported at the 0, while $U(\mathfrak{g})$ is identified with the convolution algebra of distributions on G supported at the group e. The infinitesimal $\mathfrak{g}$-action generating the left multiplication is given by the right-invariant vector fields,

$$\zeta \mapsto -\zeta^R.$$

Push-forward $\exp_*$ of distributions gives an isomorphism of distributions supported at 0 with those supported at e, and this isomorphism is exactly the symmetrization map.

5.2. The formal vector fields $X^\zeta(\phi)$. We have to develop a technique for calculating the commutator of formal vector fields of the form $e^\xi \mapsto \phi(\mathrm{ad}_\xi)\zeta$, for formal power series $\phi \in \mathbb{K}[[z]]$. To this end, we introduce the following notations. For $\zeta \in \mathfrak{g}$, we define a linear map

$$\mathbb{K}[[z]] \to \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g}), \quad \phi \mapsto X^\zeta(\phi)$$

by $X^\zeta(\phi)(e^\xi) = \phi(\mathrm{ad}_\xi)(\zeta)$. We also define linear maps

$$\mathbb{K}[[z_1, z_2]] \to \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g}), \ \psi \mapsto X^{\zeta_1,\zeta_2}(\psi)$$

for $\zeta_1, \zeta_2 \in \mathfrak{g}$, taking a monomial $z_1^{k_1} z_2^{k_2}$ to the formal vector field

$$X^{\zeta_1,\zeta_2}(z_1^{k_1} z_2^{k_2})(e^\xi) = [\mathrm{ad}_\xi^{k_1}(\zeta_1), \mathrm{ad}_\xi^{k_2}(\zeta_2)].$$

PROPOSITION 5.2. *For all $\phi \in \mathbb{K}[[z]]$ and $\zeta_1, \zeta_2 \in \mathfrak{g}$,*

$$X^{[\zeta_1,\zeta_2]}(\phi) = X^{\zeta_1,\zeta_2}(\Delta\phi)$$

where $\Delta\phi \in \mathbb{K}[[z_1, z_2]]$ is the formal power series $(\Delta\phi)(z_1, z_2) = \phi(z_1 + z_2)$.

PROOF. It is enough to check on monomials $\phi(z) = z^n$. By induction on n,

$$\mathrm{ad}_\xi^n[\zeta_1, \zeta_2] = \sum_{i=0}^n \binom{n}{i} [\mathrm{ad}_\xi^i \zeta_1, \mathrm{ad}_\xi^{n-i} \zeta_2],$$

hence $X^{[\zeta_1,\zeta_2]}(z^n) = X^{\zeta_1,\zeta_2}\big((z_1 + z_2)^n\big)$. □

Given $\phi \in \mathbb{K}[[z]]$ we define power series in two variables by

$$\delta^{(1)}\phi(z_1, z_2) = \frac{\phi(z_1 + z_2) - \phi(z_2)}{z_1}$$
$$\delta^{(2)}\phi(z_1, z_2) = \frac{\phi(z_1 + z_2) - \phi(z_1)}{z_2}.$$

PROPOSITION 5.3. *For any $\phi \in \mathbb{K}[[z]]$, and any $\zeta, Y \in \mathfrak{g}$,*

$$X^\zeta(\phi) \circ Y = X^{Y,\zeta}(\delta^{(1)}\phi), \qquad \zeta, Y \in \mathfrak{g}.$$

On the left hand side Y is identified with the operator $S(\mathfrak{g}) \to S(\mathfrak{g})$ of multiplication by Y. Put differently, we think of Y as a 'constant' element of $\mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g}) \cong \mathrm{Coder}(S(\mathfrak{g}))$.

PROOF. On monomials z^n, the formula says that

$$X^\zeta(z^n)(Ye^\xi) = X^{Y,\zeta}\Big(\frac{(z_1 + z_2)^n - z_2^n}{z_1}\Big)(e^\xi).$$

The proof is an induction on n: The cases $n=1$ is clear, while

$$\begin{aligned}
X^{\zeta}(z^{n+1})(Ye^{\xi}) &= \frac{\partial}{\partial s}\Big|_{s=0} \mathrm{ad}^{n+1}(\xi+sY)\zeta \\
&= [Y, \mathrm{ad}^n(\xi)\zeta] + \mathrm{ad}(\xi)\frac{\partial}{\partial s}\Big|_{s=0} \mathrm{ad}^n(\xi+sY)\zeta \\
&= [Y, \mathrm{ad}^n(\xi)\zeta] + \mathrm{ad}(\xi)X^{\zeta}(z^n)(Ye^{\xi}) \\
&= X^{Y,\zeta}(z_2^n)(e^{\xi}) + \mathrm{ad}(\xi)X^{Y,\zeta}\Big(\frac{(z_1+z_2)^n - z_2^n}{z_1}\Big)(e^{\xi}) \\
&= X^{Y,\zeta}\Big(z_2^n + (z_1+z_2)\frac{(z_1+z_2)^n - z_2^n}{z_1}\Big)(e^{\xi}) \\
&= X^{Y,\zeta}\Big(\frac{(z_1+z_2)^{n+1} - z_2^{n+1}}{z_1}\Big)(e^{\xi}),
\end{aligned}$$

using the induction hypothesis for the fourth equality sign. □

PROPOSITION 5.4. *The Lie bracket of vector fields $X^{\zeta_1}(\phi_1)$ and $X^{\zeta_2}(\phi_2)$ is given by the formula,*

$$[X^{\zeta_1}(\phi_1), X^{\zeta_2}(\phi_2)] = -X^{\zeta_1,\zeta_2}\big(\delta^{(2)}\phi_1\ \pi_2^*\phi_2 + \delta^{(1)}\phi_2\ \pi_1^*\phi_1\big),$$

where $(\pi_j^*\phi)(z_1,z_2) = \phi(z_j)$ for $j=1,2$.

PROOF. By definition of the Lie bracket (66),

$$[X^{\zeta_1}(\phi_1), X^{\zeta_2}(\phi_2)](e^{\xi}) = X^{\zeta_1}(\phi_1)(e^{\xi}X^{\zeta_2}(\phi_2)(e^{\xi})) - X^{\zeta_2}(\phi_2)(e^{\xi}X^{\zeta_1}(\phi_1)(e^{\xi})).$$

To compute the first term we put $Y = X^{\zeta_2}(\phi_2)(e^{\xi}) = \phi_2(\mathrm{ad}_{\xi})\zeta_2$, and use the previous Proposition:

$$X^{\zeta_1}(\phi_1)(e^{\xi}X^{\zeta_2}(\phi_2)(e^{\xi})) = (X^{Y,\zeta_1}(\delta^{(1)}\phi_1))(e^{\xi}).$$

Writing $\delta^{(1)}\phi_1$ as a linear combination of monomials $z_1^{k_1}z_2^{k_2}$, and using the computation

$$\begin{aligned}
(X^{Y,\zeta_1}(z_1^{k_1}z_2^{k_2}))(e^{\xi}) &= [\mathrm{ad}_{\xi}^{k_1} Y, \mathrm{ad}_{\xi}^{k_2}\zeta_1] \\
&= [\mathrm{ad}_{\xi}^{k_1}\phi_2(\mathrm{ad}_{\xi})\zeta_2, \mathrm{ad}_{\xi}^{k_2}\zeta_1] \\
&= X^{\zeta_2,\zeta_1}(z_1^{k_1}z_2^{k_2}\phi_2(z_1))(e^{\xi})
\end{aligned}$$

we find

$$\begin{aligned}
X^{\zeta_1}(\phi_1)(e^{\xi}X^{\zeta_2}(\phi_2)(e^{\xi})) &= X^{\zeta_2,\zeta_1}\Big(\delta^{(1)}\phi_1\ \pi_1^*\phi_2\Big)(e^{\xi}) \\
&= -X^{\zeta_1,\zeta_2}\Big(\delta^{(2)}\phi_1\ \pi_2^*\phi_2\Big)(e^{\xi}).
\end{aligned}$$

Similarly

$$X^{\zeta_2}(\phi_2)(e^{\xi}X^{\zeta_1}(\phi_1)(e^{\xi})) = X^{\zeta_1,\zeta_2}\Big(\delta^{(1)}\phi_2\ \pi_1^*\phi_1\Big)(e^{\xi}).$$

Hence the Lie bracket is $-X^{\zeta_1,\zeta_2}\big(\delta^{(2)}\phi_1\ \pi_2^*\phi_2 + \delta^{(1)}\phi_2\ \pi_1^*\phi_1\big)$, as claimed. □

As a special case $\phi = \phi_1 = \phi_2$, we see that $[X^{\zeta_1}(\phi), X^{\zeta_2}(\phi)] = X^{\zeta_1,\zeta_2}(\psi)$ where

$$\psi(z_1, z_2) = -\frac{\phi(z_1 + z_2) - \phi(z_2)}{z_1}\phi(z_1) - \frac{\phi(z_1 + z_2) - \phi(z_1)}{z_2}\phi(z_2).$$

5.3. Proof of Petracci's theorem. Since $X^{[\zeta_1,\zeta_2]}(\phi) = X^{\zeta_1,\zeta_2}(\Delta\phi)$ we see that $\zeta \mapsto X^\zeta(\phi)$ is a Lie algebra homomorphism if and only if ϕ satisfies the functional equation,

$$\phi(z_1 + z_2) + \frac{\phi(z_1 + z_2) - \phi(z_2)}{z_1}\phi(z_1) + \frac{\phi(z_1 + z_2) - \phi(z_1)}{z_2}\phi(z_2) = 0.$$

The equation may be re-written

$$\frac{\phi(z_1 + z_2)}{z_1 + z_2} = \Big(\frac{\phi(z_1)}{z_1}\frac{\phi(z_2)}{z_2}\Big)\Big(1 + \frac{\phi(z_1)}{z_1} + \frac{\phi(z_2)}{z_2}\Big)^{-1}.$$

Suppose $\phi \in \mathbb{K}[[z]]$ is a non-zero solution of this equation. Putting $z_1 = z_2 = z$, we see that the leading term (for $z \to 0$) in the expansion of ϕ cannot be of order z^k with $k > 1$, or else the left hand side would be of order z^{k-1} while the right hand side is of order z^{2k-2}. That is, if ϕ is a non-zero solution then

$$\psi(z) = 1 + \frac{z}{\phi(z)}$$

is a well-defined element $\psi \in \mathbb{K}[[z]]$. A short calculation shows that, in terms of ψ, the functional equation is simply $\psi(z_1 + z_2) = \psi(z_1)\psi(z_2)$. The solutions are $\psi(z) = e^{cz}$ with $c \in \mathbb{K}$, together with the trivial solution $\psi(z) = 0$. We conclude that the solutions of the functional equation for ϕ are

$$\phi(z) = \frac{z}{e^{cz} - 1},\ \ c \neq 0$$

together with the solutions $\phi(z) = -z$ and $\phi(z) = 0$. In particular, there is a unique solution with $\phi(0) = 1$, given by $\phi(z) = \frac{z}{e^z - 1}$. This proves Petracci's theorem 5.1.

Remark 5.5. The $\mathfrak{g}$-representation on $S(\mathfrak{g})$ defined by the exceptional solution $\phi(z) = -z$ is just the adjoint representation:

$$C^\zeta(-z)(e^\xi) \equiv e^\xi X^\zeta(-z)(e^\xi) = [\zeta, \xi]\ e^\xi \quad \Rightarrow \quad C^\zeta(-z)(\xi^n) = n[\zeta, \xi]\, \xi^{n-1}.$$

6. The center of the enveloping algebra

We have already observed that the center of the enveloping algebra $U(\mathfrak{g})$ is just the ad-invariant subspace,

$$\mathrm{Cent}(U(\mathfrak{g})) = (U(\mathfrak{g}))^{\mathfrak{g}}.$$

Elements of the center are also called *Casimir elements*. The symmetrization map is $\mathfrak{g}$-equivariant, hence it defines an isomorphism of vector spaces $\mathrm{sym}\colon (S\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cent}(U(\mathfrak{g}))$.

EXAMPLE 6.1. Suppose that $\mathfrak{g}$ carries an invariant non-degenerate symmetric bilinear form B. If e_i is a basis of $\mathfrak{g}$, and e^i the B-dual basis (so that $B(e_i, e^j) = \delta_i^j$), the element $p = \sum_i e_i e^i \in S^2(\mathfrak{g})$ is invariant. Its image $\mathrm{sym}(p) \in U(\mathfrak{g})$ under symmetrization is called the *quadratic Casimir* of the bilinear form.

REMARK 6.2. Suppose $\mathbb{K} = \mathbb{R}$, and that $\mathfrak{g}$ is the Lie algebra of Lie group G. In terms of the identification $U(\mathfrak{g}) \cong \mathfrak{D}^L(G)$, the center corresponds to the space $\mathfrak{D}^{L\times R}(G)$ of bi-invariant differential operators. For instance, if G is a quadratic Lie group (i.e. if $\mathfrak{g}$ carries a non-degenerate invariant symmetric bilinear form B), then the operator

$$D = \sum_i e_i^L (e^i)^L$$

(where $e_i \in \mathfrak{g}$ is a basis of $\mathfrak{g}$, and $e^i \in \mathfrak{g}$ is the B-dual basis) is the bi-invariant differential operator corresponding to the quadratic Casimir element.

Suppose for the rest of this section that $\mathbb{K} = \mathbb{C}$ and $\dim \mathfrak{g} < \infty$. Suppose $\varrho\colon \mathfrak{g} \to \mathrm{End}(E)$ is a $\mathfrak{g}$-representation on a finite-dimensional complex vector space E. Extend to a representation $\varrho\colon U(\mathfrak{g}) \to \mathrm{End}(E)$. If $x \in \mathrm{Cent}(U(\mathfrak{g}))$, the operator $\varrho(x)$ commutes with all $\varrho(y)$, $y \in U(\mathfrak{g})$. If ϱ is irreducible, this implies by Schur's lemma that $\varrho(x)$ is a multiple of the identity. That is, any irreducible representation determines an algebra homomorphism

$$\mathrm{Cent}(U(\mathfrak{g})) \to \mathbb{K},\ x \mapsto \varrho(x).$$

For semi-simple Lie algebras, it is known that this algebra homomorphism characterizes ϱ up to isomorphism. In fact, it suffices to know this map on a set of generators for $\mathrm{Cent}(U(\mathfrak{g}))$. For example, if $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, any irreducible representation is determined by the value of the quadratic Casimir in this representation. It is therefore of interest to understand the structure of $\mathrm{Cent}(U(\mathfrak{g}))$ as an algebra.

The symmetrization map $\mathrm{sym}\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ restricts to an isomorphism on invariants, $(S(\mathfrak{g}))^{\mathfrak{g}} \to (U(\mathfrak{g}))^{\mathfrak{g}} = \mathrm{Cent}(U(\mathfrak{g}))$. Unfortunately this restricted map is not an algebra homomorphism.

EXAMPLE 6.3. Suppose $\mathfrak{g}$ is a quadratic Lie algebra, with non-degenerate symmetric bilinear form B. Let e_i be a basis of $\mathfrak{g}$ and e^i the B-dual basis, and consider $p = \sum_i e_i e^i$. Let $f_{ijk} = B([e_i, e_j], e_k)$ be the structure constants. We use B to raise or lower indices, e.g. ${f_{ij}}^k = \sum_l f_{ijl} B(e^l, e^m)$. We have

$$\begin{aligned} \mathrm{sym}(p^2) &= \sum_{ij} \mathrm{sym}(e_i e^i e_j e^j) \\ &= \frac{1}{3} \sum_{ij} (e_i e^i e_j e^j + e^i e_j e_i e^j + e_i e_j e^j e^i) \end{aligned}$$

where we have used basis independence to identify some of the expressions coming from the symmetrization. (For instance, $\sum_i e_i e_j e^i e^j = \sum_i e^i e_j e_i e^j$.)

Using the defining relations in the enveloping algebra, this becomes

$$\begin{aligned}
\mathrm{sym}(p^2) &= \frac{1}{3}\sum_{ij}(2e_ie^ie_je^j - e^i[e_i,e_j]e^j + e_ie_je^ie^j - e^ie^j[e_i,e_j]) \\
&= \frac{1}{3}\sum_{ij}(3e_ie^ie_je^j - 2e^i[e_i,e_j]e^j - e^ie^j[e_i,e_j]) \\
&= (\mathrm{sym}(p))^2 + \frac{1}{3}\sum_{ijk}(f_{ijk}e^ie^je^k) \\
&= (\mathrm{sym}(p))^2 + \frac{1}{6}\sum_{ijkl}(f_{ijk}f^{ijl}e_le^k)
\end{aligned}$$

We hence see that $\mathrm{sym}(p^2) \neq (\mathrm{sym}(p))^2$ in general. Note that $\sum_{ij}(f_{ijk}f^{ijl}$ are the coefficients of the Killing form on $\mathfrak{g}$. If $\mathfrak{g}$ is simple, the Killing form is a multiple of B, and the correction term is a multiple of $\mathrm{sym}(p)$. For instance, if $\mathfrak{g} = \mathfrak{so}(3)$, with B-orthornormal basis e_1, e_2, e_3 satisfying $[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2$, we obtain

$$\mathrm{sym}(p^2) = \mathrm{sym}(p)^2 + \frac{1}{3}\mathrm{sym}(p).$$

Duflo's theorem says that this failure of $\mathrm{sym}\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ to restrict to an algebra isomorphism on invariants, can be repaired by pre-composing sym with a suitable infinite-order differential operator on $S(\mathfrak{g})$.

Let us introduce such infinite-order differential operators on $S(V)$ for any finite-dimensional vector space V. The symmetric algebra $S(V)$ is identified with the algebra of polynomials on V^*. Any $\mu \in V^*$ defines a derivation $\iota_S(\mu)$ of this algebra, given by

$$\iota_S(\mu)v = \langle \mu, v\rangle$$

on generators $v \in V$. The map $\mu \mapsto \iota_S(\mu)$ extends to an algebra homomorphism $S(V^*) \to \mathrm{End}(S(V))$, and even further to an algebra homomorphism $S(V)^* \to \mathrm{End}(S(V))$ where $S(V)^* = \overline{S}(V^*)$ is the algebraic dual

$$S(V)^* = \prod_{k=0}^{\infty} S^k(V^*).$$

One may think of $S(V)^*$ as an algebra of 'infinite order differential operators' acting on polynomials $S(V) = \mathrm{Pol}(V^*)$.

For $p \in S(V)^*$, the corresponding operator $\widetilde{p} = \iota_S(p) \in \mathrm{End}(S(V))$, is characterized by the equation

$$\widetilde{p}(e^{tv}) = p(tv)e^{tv},$$

an equality of formal power series in t with coefficients in $S(V)$.

More generally, if f is a smooth function defined on some open neighborhood of $0 \in V$, its Taylor series expansion is an element of $S(V)^*$, and hence the operator $\widetilde{f} \in \mathrm{End}(S(V))$ is well-defined.

Returning to the setting for Duflo's theorem, consider the function

$$J^{1/2}(\xi) = \det{}^{1/2}(j(\mathrm{ad}_\xi)) = e^{1/2\,\mathrm{tr}\log(j(\mathrm{ad}_\xi))},$$

with $j(z) = \frac{\sinh(z/2)}{z/2}$. As discussed in §4.3, $J^{1/2}$ is a well-defined holomorphic function of $\xi \in \mathfrak{g}$ if $\mathbb{K} = \mathbb{C}$. Modulo terms of order ≥ 4 in ξ, one finds

$$J^{1/2}(\xi) = 1 + \frac{1}{48}\,\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_\xi^2) + \ldots.$$

Note that the first correction term is a multiple of the quadratic form associated to the Killing form on $\mathfrak{g}$. If $\mathbb{K}$ is a arbitrary field of characteritic zero, $J^{1/2}$ is is well-defined (via its Taylor series) as an element

$$J^{1/2} \in \mathrm{Hom}(S(\mathfrak{g}), \mathbb{K}) = S(\mathfrak{g})^*$$

The function $J^{1/2}$ defines an operator $\widetilde{J^{1/2}} \in \mathrm{End}(S(\mathfrak{g}))$.

THEOREM 6.4 (Duflo [**25**]). *The composition*

$$\mathrm{sym} \circ \widetilde{J^{1/2}} \colon S(\mathfrak{g}) \to U(\mathfrak{g})$$

restricts to an algebra isomorphism $(S(\mathfrak{g}))^{\mathfrak{g}} \to \mathrm{Cent}(U(\mathfrak{g}))$.

In §7.3, we present a proof of Duflo's theorem for the case that $\mathfrak{g}$ is a quadratic Lie algebra. This proof will relate the appearance of the factor $J^{1/2}$ in Duflo's theorem with that in the theory of Clifford algebras.

CHAPTER 6

Weil algebras

For any Lie algebra $\mathfrak{g}$, the exterior and symmetric algebras over $\mathfrak{g}^*$ combine into a differential algebra $W(\mathfrak{g})$, the *Weil algebra*. The present chapter will develop some aspects of the theory of differential spaces and $\mathfrak{g}$-differential spaces, as needed for our purposes. (We refer to Appendix A to background material on graded and filtered super vector spaces.) We will then introduce the Weil algebra as a universal object among commutative $\mathfrak{g}$-differential algebras with connection. By considering *non-commutative* $\mathfrak{g}$-differential algebras with connection, we are led to introduce also a *non-commutative Weil algebra*. We will discuss applications of the two Weil algebras to Chern-Weil theory and to transgression.

1. Differential spaces

DEFINITION 1.1. A *differential space* is a super vector space $E = E^{\bar{0}} \oplus E^{\bar{1}}$, equipped with an odd operator $\mathrm{d}\colon E \to E$ such that $\mathrm{d} \circ \mathrm{d} = 0$, i.e. $\mathrm{im}(\mathrm{d}) \subset \ker(\mathrm{d})$. One calls

$$H(E, \mathrm{d}) = \frac{\ker(\mathrm{d})}{\mathrm{im}(\mathrm{d})}$$

the *cohomology* of the differential space (E, d). It is again a super vector space, with $\mathbb{Z}_2$-grading inherited from E. A *morphism of differential spaces* (also called *cochain map*) $(E_1, \mathrm{d}_1) \to (E_2, \mathrm{d}_2)$ is a morphism of super vector spaces intertwining the differentials.

The category of differential spaces, with cochain maps as morphisms, has direct sums

$$(E_1, \mathrm{d}_1) \oplus (E_2, \mathrm{d}_2) = (E_1 \oplus E_2,\ \mathrm{d}_1 \oplus \mathrm{d}_2)$$

and tensor products

$$(E_1, \mathrm{d}_1) \otimes (E_2, \mathrm{d}_2) = (E_1 \otimes E_2,\ \mathrm{d}_1 \otimes 1 + 1 \otimes \mathrm{d}_2),$$

with the usual compatibility properties. It is thus a *tensor category*, and one can consider its algebra objects, Lie algebra objects and so on. For example, a *differential algebra* $(\mathcal{A}, \mathrm{d})$ is a differential space with a multiplication morphism

$$m\colon (\mathcal{A}, \mathrm{d}) \otimes (\mathcal{A}, \mathrm{d}) \to (\mathcal{A}, \mathrm{d})$$

and a unit morphism

$$i\colon (\mathbb{K}, 0) \to (\mathcal{A}, \mathrm{d})$$

satisfying the algebra axioms (Equation (65) in §5.4.1). One finds that this is equivalent to $\mathcal{A}$ being a super algebra, with a differential d that is a *derivation* of the product. The cohomology $H(\mathcal{A}, \mathrm{d})$ of a differential algebra is again a super algebra. One similarly defines differential Lie algebras, differential coalgebras, differential Hopf algebras and so on. We will not spell out all of these definitions.

There are parallel definitions of categories of *graded differential spaces* (also known as *cochain complexes*) or *filtered differential spaces*. Thus, a graded (resp. filtered) differential space (E, d) is a graded (resp. filtered) super vector space E with a differential d of degree (resp. filtration degree) 1. Its cohomology $H(E, \mathrm{d})$ is again a graded (resp. filtered) super vector space. Morphisms of graded (resp. filtered) differential spaces are morphism of graded (resp. filtered) super vector spaces intertwining the differentials.

Remark 1.2. If (E, d) is a (graded, filtered) differential space, and $n \in \mathbb{Z}$ then the same space with degree shift $(E[n], \mathrm{d})$ is a (graded, filtered) differential space $(E[n], \mathrm{d})$. Indeed, we can regard $E[n]$ as a tensor product of differential spaces $E \otimes \mathbb{K}[n]$ where $\mathbb{K}[n]$ carries the zero differential.

Example 1.3. For any manifold M, the algebra of differential forms $\Omega(M)$ is a graded differential algebra.

Example 1.4. Let $\mathbb{K}[\iota]$ be the commutative graded super algebra[1], generated by an element ι of degree -1 satisfying $\iota^2 = 0$. (Thus $\mathbb{K}[\iota] = \mathbb{K} \cdot \iota \oplus \mathbb{K}$ as a graded super vector space.) Then $\mathbb{K}[\iota]$ is a graded differential algebra for the differential $\mathrm{d}(b\iota + a) = b$.

Example 1.5. Let $\mathfrak{g}$ be a Lie algebra, and consider the graded super Lie algebra $\mathfrak{g}[1] \rtimes \mathfrak{g}$ with degree -1 generators $I_\xi \in \mathfrak{g}[1]$ and degree 0 generators $L_\xi \in \mathfrak{g}$, labeled by $\xi \in \mathfrak{g}$. It is a graded differential Lie algebra with differential is $\mathrm{d}(I_\xi) = L_\xi,\ \mathrm{d}(L_\xi) = 0$. It may also be viewed as follows. Regard $\mathfrak{g}$ as a graded differential Lie algebra with trivial grading and zero differential. Then

$$\mathfrak{g}[1] \rtimes \mathfrak{g} = \mathfrak{g} \otimes \mathbb{K}[\iota] = \mathfrak{g}[\iota],$$

a tensor product with the commutative graded differential algebra $\mathbb{K}[\iota]$ from Example 1.4.

2. Symmetric and tensor algebra over differential spaces

Suppose E is a super vector space. Then the tensor algebra $T(E)$ and the symmetric algebra $S(E)$ carry the structure of super algebras, in such a way that the inclusion of E is a morphism of super vector spaces. Rcall that the definition of the symmetric algebra $S(E)$ uses the super-sign convention (cf. Appendix A): It is the algebra with generators $v \in E$, and relations

$$vw - (-1)^{|v||w|} wv = 0,$$

[1]Here $[\cdot]$ does not indicate a degree shift, but signifies a polynomial ring.

for homogeneous elements $v, w \in E$.

If E is a graded (resp. filtered) super space, the algebras $S(E), T(E)$ inherit an *internal grading* (resp. *internal filtration*), with the property that the inclusion of E preserves degrees. We can also consider the *total grading* (resp. *total filtration*), obtained by adding twice the external degree, i.e. such that the inclusion defines morphisms of graded resp. filtered super spaces $E[-2] \to S(E), T(E)$. Note that $S(E)$ with the total grading is isomorphic to $S(E[-2])$ with the internal grading, and similarly for the tensor algebra.

From now on, unless specified otherwise, we will always work with the internal grading or filtration.

If (E, d) is a (graded, filtered) differential space, then $S(E)$, $T(E)$ carry the structure of (graded, filtered) differential algebras, in such a way that the inclusion of E is a morphism. The differential on these super algebras is the derivation extension of the differential on E; the property $[\mathrm{d}, \mathrm{d}] = 0$ is immediate since it holds on generators. Similarly, if $(\mathfrak{g}, \mathrm{d})$ is a (graded, filtered) differential Lie algebras, thn the enveloping algebra $U(\mathfrak{g})$ is a (graded, filtered) differential algebra.

3. Homotopies

Let $E_{\mathbb{K}} = \mathbb{K}[0] \oplus \mathbb{K}[-1]$, where $\mathbb{K}[0]$ is spanned by a generator t and $\mathbb{K}[-1]$ by a generator $\bar{t}$. Then $E_{\mathbb{K}}$ is a graded differential space, with differential

$$\mathrm{d}t = \bar{t}, \quad \mathrm{d}\bar{t} = 0.$$

The symmetric algebra over $E_{\mathbb{K}}$ is the commutative graded differential algebra $S(E_{\mathbb{K}}) = \mathbb{K}[t, \mathrm{d}t]$, with generators t of degree 0 and $\mathrm{d}t$ of degree 1 and with the single relation $(\mathrm{d}t)^2 = 0$. A general element of this algebra is a finite linear combination

$$y = \sum_k a_k t^k + \sum_l b_l t^l \mathrm{d}t \tag{67}$$

with $a_k, b_l \in \mathbb{K}$. Let $\pi_0, \pi_1 \colon \mathbb{K}[t,\ dt] \to \mathbb{K}$ be the morphisms of differential algebras, given on the element (67) by

$$\pi_0(y) = a_0, \quad \pi_1(y) = \sum_k a_k.$$

One can think of $\mathbb{K}[t, \mathrm{d}t]$ as an algebraic counterpart to differential forms on a unit interval, with π_0, π_1 the evaluations at the end points. In the same spirit, we can define an "integration operator" $J \colon \mathbb{K}[t, \mathrm{d}t] \to \mathbb{K}$,

$$J\big(\sum_k a_k t^k + \sum_l b_l t^l \mathrm{d}t\big) = \sum_l \frac{b_l}{l+1}.$$

This satisfies

Lemma 3.1 (Stokes' formula). *The integration operator $J \colon \mathbb{K}[t, dt] \to \mathbb{K}$ has the property $J \circ d = \pi_1 - \pi_0$.*

PROOF. For $y = \sum_{k\geq 0} a_k t^k + \sum_{l\geq 0} b_l t^l \mathrm{d}t$,

$$J(\mathrm{d}y) = J(\sum_{k>0} k a_k t^{k-1}\mathrm{d}t) = \sum_{k>0} a_k = (\pi_1 - \pi_0)(y). \qquad \square$$

DEFINITION 3.2. A *homotopy* between two morphisms $\phi_0, \phi_1 \colon E \to E'$ of (graded, filtered) differential spaces $(E, \mathrm{d}), (E', \mathrm{d}')$ is a morphism

$$\phi \colon E \to \mathbb{K}[t, \mathrm{d}t] \otimes E'$$

such that $\phi_0 = (\pi_0 \otimes 1) \circ \phi$ and $\phi_1 = (\pi_1 \otimes 1) \circ \phi$. In this case, ϕ_0, ϕ_1 are called *homotopic*.

For instance, the two projections $\pi_0, \pi_1 \colon \mathbb{K}[t, \mathrm{d}t] \to \mathbb{K}$ are homotopic: the identity morphism of $\mathbb{K}[t, \mathrm{d}t]$ gives a homotopy. Homotopies can be composed: If

$$\phi \colon E \to \mathbb{K}[t, \mathrm{d}t] \otimes E', \quad \psi \colon E' \to \mathbb{K}[t, \mathrm{d}t] \otimes E''$$

are homotopies between $\phi_0, \phi_1 \colon E \to E'$ and $\psi_0, \psi_1 \colon E' \to E''$, respectively, then

$$(1 \otimes \psi) \circ \phi \colon E \to \mathbb{K}[t, \mathrm{d}t] \otimes \mathbb{K}[t, \mathrm{d}t] \otimes E'',$$

followed by multiplication in $\mathbb{K}[t, \mathrm{d}t]$, is a homotopy $\psi * \phi$ between $\psi_0 \circ \phi_0$ and $\psi_1 \circ \phi_1$. Note that the composition $*$ is associative. Let us write $\phi_0 \sim \phi_1$ for the relation of homotopy of morphisms of differential spaces.

PROPOSITION 3.3. *The relation $\sim$ of homotopy of cochain maps between differential spaces E, E' is an equivalence relation.*

PROOF. If $\phi_0 \colon E \to E'$ is a cochain map, then $\phi_0 \sim \phi_0$ by the homotopy $\phi = i \otimes \phi_0 \colon E = \mathbb{K} \otimes E \to \mathbb{K}[t, \mathrm{d}t] \otimes E'$, where $i \colon \mathbb{K} \to \mathbb{K}[t, \mathrm{d}t]$ is the inclusion of scalars. Hence $\sim$ is reflexive. Too see that it is symmetric, note that the graded differential algebra $\mathbb{K}[t, \mathrm{d}t]$ carries an involution, given on generators by $t \mapsto 1 - t,\ \mathrm{d}t \mapsto -\mathrm{d}t$. This involution intertwines the two morphisms $\pi_0, \pi_1 \colon \mathbb{K}[t, \mathrm{d}t] \to \mathbb{K}$. Hence, if $\phi_0 \sim \phi_1 \colon E \to E'$ by the homotopy $\phi \colon E \to \mathbb{K}[t, \mathrm{d}t] \otimes E'$, then $\phi_1 \sim \phi_0$ by the composition of ϕ with this involution. Finally, since one can add and subtract homotopies, $\phi_0 \sim \phi_1 \colon E \to E'$ and $\phi_1 \sim \phi_2 \colon E \to E'$ imply

$$\phi_0 = \phi_0 + \phi_1 - \phi_1 \sim \phi_1 + \phi_2 - \phi_1 = \phi_2.$$

This shows that $\sim$ is transitive. $\square$

Homotopies are often expressed in terms of homotopy operators.

DEFINITION 3.4. A *homotopy operator* between $\phi_0, \phi_1 \colon E \to E'$ of differential spaces $(E, \mathrm{d}), (E', \mathrm{d}')$ is an odd linear map $h \colon E \to E'$ with

$$h \circ \mathrm{d} + \mathrm{d}' \circ h = \phi_1 - \phi_0.$$

In the graded (resp. filtered) case, one requires that h has degree (resp. filtration degree) -1.

For instance, the integration operator $J\colon \mathbb{K}[t, \mathrm{d}t] \to \mathbb{K}$ is a homotopy operator between π_0, π_1.

PROPOSITION 3.5. *Two morphisms $\phi_0, \phi_1\colon E \to E'$ of differential spaces are homotopic if and only if there exists a homotopy operator. In this case, the induced maps in cohomology are the same: $H(\phi_0) = H(\phi_1)$.*

PROOF. A homotopy operator h defines a homotopy $\phi\colon E \to \mathbb{K}[t, \mathrm{d}t]\otimes E'$ by

$$\phi(x) = \mathrm{d}t\ h(x) + \ t\phi_1(x) + (1-t)\phi_0(x).$$

Conversely, given a homotopy ϕ between ϕ_0, ϕ_1, the formula

$$h = (J \otimes 1) \circ \phi$$

defines a homotopy operator. The equation for the homotopy operator shows in particular that $\phi_0 - \phi_1$ takes $\ker(\mathrm{d})$ to $\mathrm{im}(\mathrm{d}')$. That is, $H(\phi_0) - H(\phi_1) = H(\phi_0 - \phi_1) = 0$. $\square$

DEFINITION 3.6. Two morphisms $\phi\colon E \to E'$ and $\psi\colon E' \to E$ are called *homotopy inverses* if

$$\phi \circ \psi \sim \mathrm{id}_{E'}, \quad \psi \circ \phi \sim \mathrm{id}_E$$

A morphism ϕ admitting a homotopy inverse ψ is also called a *homotopy equivalence.*

In this case, $H(\phi)$ induces an isomorphism in cohomology, with inverse $H(\psi)$.

EXAMPLE 3.7. Let $i\colon \mathbb{K} \to \mathbb{K}[t, \mathrm{d}t]$ be the inclusion of scalars, and $\pi\colon \mathbb{K}[t, \mathrm{d}t] \to \mathbb{K}$ the augmentation map $\epsilon(\sum_k a_k t^k + \sum_l b_l t^l \mathrm{d}t) = a_0$. Then $\epsilon \circ i = \mathrm{id}_{\mathbb{K}}$. Let h be 'integration from 0 to t',

$$h\big(\sum_k a_k t^k + \sum_l b_l t^l \mathrm{d}t\big) = \sum_l \frac{b_l}{l+1} t^{l+1}.$$

Then $h\mathrm{d} + \mathrm{d}h = \mathrm{id} - i \circ \epsilon$, showing that i is a homotopy equivalence, with ϵ its homotopy inverses. That is, the differential algebra $\mathbb{K}[t, \mathrm{d}t]$ is acyclic.

4. Koszul algebras

Given a (graded, filtered) super vector space V, we obtain a (graded, filtered) differential space $E_V = V \otimes E_{\mathbb{K}}$. For $v \in V$, we will write $v \otimes t =: v$ and $v \otimes \bar{t} =: \overline{v}$; hence $E_V = V \oplus V[-1]$ with differential $\mathrm{d}v = \overline{v}$, $\mathrm{d}\overline{v} = 0$. It is characterized by the universal property that if E is a (graded filtered) differential space, then any morphism $V \to E$ of (graded, filtered) super spaces extends uniquely to a morphism $E_V \to E$ of (graded, filtered) differential spaces. Note that E_V has zero cohomology, since $\mathrm{im}(\mathrm{d}) = V[-1] = \ker(\mathrm{d})$.

DEFINITION 4.1. The differential algebra $S(E_V)$ will be called the *Koszul algebra* for the (graded, filtered) super vector space V.

It is characterized by a universal property:

PROPOSITION 4.2 (Universal property of Koszul algebra). *For any commutative (graded, filtered) differential algebra* $(\mathcal{A}, d)$*, and any morphism of (graded, filtered) super vector spaces* $V \to \mathcal{A}$*, there is a unique extension to a homomorphism of (graded, filtered) differential algebras* $S(E_V) \to \mathcal{A}$*.*

PROOF. The morphism $V \to \mathcal{A}$ of super vector spaces extends uniquely to a morphism $E_V = V \oplus V[-1] \to \mathcal{A}$ of differential spaces. In turns, by the universal property of symmetric algebras (Appendix A, Proposition 1.3) it extends further to a morphism $S(E_V) \to \mathcal{A}$ of super algebras; it is clear that this morphism intertwines differentials. □

One can also consider a *non-commutative version* of the Koszul algebra, using the tensor algebra $T(E_V)$ rather than the symmetric algebra. Using the same argument as for $S(E_V)$, we find:

PROPOSITION 4.3 (Universal property of non-commutative Koszul algebra). *For any (graded, filtered) differential algebra* $(\mathcal{A}, d)$ *and any morphism of (graded, filtered) super vector spaces* $V \to \mathcal{A}$*, there is a unique extension to a morphism of (graded, filtered) differential algebras* $T(E_V) \to \mathcal{A}$*.*

The Koszul algebras $S(E)$, $T(E)$ play the role of 'contractible spaces' in the category of commutative, non-commutative differential algebras. In fact one has:

THEOREM 4.4. *Let* $\mathcal{A}$ *be a (graded, filtered) differential algebra. Then any two morphisms of (graded, filtered) differential algebras* $\phi_0, \phi_1 \colon T(E_V) \to \mathcal{A}$ *are homotopic. If* $\mathcal{A}$ *is commutative, then any two homomorphisms of (graded, filtered) differential algebras* $\phi_0, \phi_1 \colon S(E_V) \to \mathcal{A}$ *are homotopic.*

PROOF. We present the proof for $T(E_V)$. (The proof for $S(E_V)$ is parallel.) Define a linear map

$$\phi \colon V \to \mathbb{K}[t, \mathrm{d}t] \otimes \mathcal{A}, \quad \phi(v) = (1-t)\phi_0(v) + t\phi_1(v),$$

and extend to a morphism of differential algebras

$$\phi \colon T(E_V) \to \mathbb{K}[t, \mathrm{d}t] \otimes \mathcal{A}$$

by the universal property of the Koszul algebra. Then $\phi_i = (\pi_i \otimes 1) \circ \phi$, since both sides are morphisms of differential algebras $T(E_V) \to \mathcal{A}$ that agree on V. □

COROLLARY 4.5. *The non-commutative Koszul algebra* $T(E_V)$ *is* acyclic. *That is, the augmentation map* $\epsilon \colon T(E_V) \to \mathbb{K}$ *and the unit* $i \colon \mathbb{K} \to T(E_V)$ *are homotopy inverses. Similarly the commutative Koszul algebra* $S(E_V)$ *is acyclic.*

PROOF. We have to show that $i \circ \epsilon \colon T(E_V) \to T(E_V)$ is homotopic to $\mathrm{id} \colon T(E_V) \to T(E_V)$. But according to Theorem 4.4, *any* two morphisms of differential algebras $T(E_V) \to T(E_V)$ are homotopic. The proof for $S(E_V)$ is parallel. □

In particular, the unit maps $i\colon \mathbb{K} \to S(E_V),\ T(E_V)$ define isomorphisms of the cohomology algebras of $S(E_V),\ T(E_V)$ with $\mathbb{K}$.

Remark 4.6. Suppose that the (graded, filtered) super vector space V carries a representation of a Lie algebra $\mathfrak{g}$. Then the action of $\mathfrak{g}$ on E_V commutes with the differential, and so does its derivation extension to $S(E_V)$ and to $T(E_V)$. The homotopy equivalences considered above are all $\mathfrak{g}$-equivariant, In particular, it follows that the $\mathfrak{g}$-invariant parts $S(E_V)^{\mathfrak{g}}$ and $T(E_V)^{\mathfrak{g}}$ are acyclic.

5. Symmetrization

Suppose E is a (graded, filtered) super vector space, $\mathcal{A}$ is a (graded filtered) super algebra (not necessarily commutative), and $\phi\colon E \to \mathcal{A}$ is a morphism of (graded, filtered) super vector spaces. Then ϕ extends canonically to a morphism of (graded, filtered) super vector spaces

$$\mathrm{sym}(\phi)\colon S(E) \to \mathcal{A},$$

by super symmetrization: For homogeneous elements $v_i \in E$,

$$v_1 \cdots v_k \mapsto \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} (-1)^{N_s(v_1,\ldots,v_k)} \phi(v_{s^{-1}(1)}) \cdots \phi(v_{s^{-1}(k)}).$$

Here $N_s(v_1, \ldots, v_k)$ is the number of pairs $i < j$ such that v_i, v_j are odd elements and $s^{-1}(i) > s^{-1}(j)$. (The sign is dictated by the super sign convention.) If the super algebra $\mathcal{A}$ is commutative, this is the unique extension as a morphism of super algebras. For the special case $\mathcal{A} = T(E)$, the symmetrization map is the inclusion as 'symmetric tensors', and the general case may be viewed as this inclusion followed by the algebra homomorphism $T(E) \to \mathcal{A}$.

The symmetrization map can also be characterized as follows. Let e_i be a homogeneous basis of E, and let τ^i be formal parameters, with τ^i even or odd depending on whether e_i is even or odd. Then

$$\mathrm{sym}(\phi)\big(\sum_i \tau^i e_i\big)^k = \big(\sum_i \tau^i \phi(e_i)\big)^k$$

for all k. Equivalently

$$\mathrm{sym}(\phi)\big(e^{\sum_i \tau^i e_i}\big) = e^{\sum_i \tau^i \phi(e_i)},$$

as an equality of formal power series in the τ_i.

Lemma 5.1. *Let E be a super vector space, and $S(E), T(E)$ the symmetric and tensor algebras respectively. Let D_S, D_T be the derivation extensions of a given (even or odd) endomorphism of $D \in \mathrm{End}(E)$. Then the inclusion $S(E) \to T(E)$ as symmetric tensors intertwines D_S, D_T.*

Proof. This follows since the action of D_T on $T^k(E)$ commutes with the action of the symmetric group $\mathfrak{S}_k$, and in particular preserves the invariant subspace. For $k = 2$, with $\mathcal{T} \in \mathfrak{S}_2$ the transposition, this is checked by the computation

$$\begin{aligned}\mathcal{T} D_T(v_1 \otimes v_2) &= \mathcal{T}(Dv_1 \otimes v_2 + (-1)^{|D||v_1|} v_1 \otimes Dv_2 \\ &= (-1)^{|v_1||v_2|+|D||v_2|} v_2 \otimes Dv_1 + (-1)^{|v_1||v_2|} Dv_2 \otimes v_1 \\ &= (-1)^{|v_1||v_2|} D_T(v_2 \otimes v_1) \\ &= D_T \mathcal{T}(v_1 \otimes v_2).\end{aligned}$$

The general case is reduced to the case $k = 2$, by writing a general element of $\mathfrak{S}_k$ as a product of transpositions. ☐

Proposition 5.2. *Suppose $\mathcal{A}$ is a differential algebra, E a differential space, and $\phi\colon E \to \mathcal{A}$ is a morphism of differential spaces. Then the symmetrized map* $\operatorname{sym}(\phi)\colon S(E) \to \mathcal{A}$ *is a morphism of differential spaces.*

Proof. By Lemma 5.1, the inclusion $S(E) \to T(E)$ is a morphism of differential spaces, hence so is its composition with $T(E) \to \mathcal{A}$. ☐

As a special case, if V is a super vector space, the symmetrization map for Koszul algebras $S(E_V) \to T(E_V)$ is a morphism of differential spaces.

Proposition 5.3. *The quotient map $\pi\colon T(E_V) \to S(E_V)$ is a homotopy equivalence of graded differential spaces, with homotopy inverse given by symmetrization* $\operatorname{sym}\colon S(E_V) \to T(E_V)$.

Proof. Since $\pi \circ \operatorname{sym}$ is the identity, it suffices to show that $\operatorname{sym} \circ \pi$ is homotopic to the identity of $T(E_V)$. Let $\phi_0, \phi_1\colon T(E_V) \to T(E_V) \otimes S(E_V)$ be the two morphisms of differential algebras $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$. By Theorem 4.4, ϕ_0, ϕ_1 are homotopic in the category of graded differential spaces. Let $\psi\colon T(E_V) \otimes S(E_V) \to T(E_V),\ x \otimes y \mapsto x \operatorname{sym}(y)$. Then ψ is a morphism of graded differential spaces. We have

$$\psi \circ \phi_0 = \mathrm{id}_{T(E_V)}, \quad \psi \circ \phi_1 = \operatorname{sym} \circ \pi.$$

☐

6. $\mathfrak{g}$-differential spaces

Differential algebras may be thought of as a generalization of differential forms on manifolds. Viewed in this way, the $\mathfrak{g}$-differential algebras discussed below are a generalization of differential manifolds on a manifolds with a $\mathfrak{g}$-action. The concept of a $\mathfrak{g}$-differential algebra was introduced by H. Cartan in [**18, 19**].

Let $\mathfrak{g}$ be a Lie algebra. Recall the graded differential Lie algebra $\mathfrak{g}[1] \rtimes \mathfrak{g}$ introduced in Example 1.5.

Definition 6.1. A (graded, filtered) $\mathfrak{g}$-*differential space* is a (graded, filtered) differential space (E, d), together with a representation of the (graded, filtered) differential Lie algebra $\mathfrak{g}[1] \rtimes \mathfrak{g}$.

Here, 'representation' is understood in the category of (graded, filtered) differential spaces: In particular, the action map $(\mathfrak{g}[1] \rtimes \mathfrak{g}) \otimes E \to E$ intertwines the differentials. Letting $\iota(\xi) \in \mathrm{End}^{\bar{1}}(E)$, $L(\xi) \in \mathrm{End}^{\bar{0}}(E)$ be the images of $I_\xi, L_\xi \in \mathfrak{g}[1] \rtimes \mathfrak{g}$ under the representation, we find that the axioms of a $\mathfrak{g}$-differential space are equivalent to the following super commutator relations,

$$\begin{aligned}
[\mathrm{d}, \mathrm{d}] &= 0 \\
[\iota(\xi), \mathrm{d}] &= L(\xi) \\
[L(\xi), \mathrm{d}] &= 0 \\
[L(\xi), L(\zeta)] &= L([\xi, \zeta]) \\
[L(\xi), \iota(\zeta)] &= \iota([\xi, \zeta]) \\
[\iota(\xi), \iota(\zeta)] &= 0
\end{aligned}$$

For a *graded (resp. filtered) $\mathfrak{g}$-differential space*, the operators d, $L(\xi)$, $\iota(\xi)$ have degree (resp. filtration degree) $1, 0, -1$.

Example 6.2. If V is a (graded, filtered) vector space with a representation $\varrho\colon \mathfrak{g} \to \mathrm{End}(V)$, then the differential space structure of $E_V = V \oplus V[-1]$ extends uniquely to a that of a (graded, filtered) $\mathfrak{g}$-differential space, in such a way that $L(\xi)v$ for $v \in V$ is the given representation and $\iota(\xi)v = 0$. (The formulas on $\overline{v} \in V[-1]$ are determined by the commutator relations.)

Remark 6.3. View $\mathbb{K}[-1]$ as a commutative graded super Lie algebra, with generator D of degree 1. A (graded, filtered) differential space may be regarded as a representation of $\mathbb{K}[-1]$, in the category of (graded, filtered) super spaces, with D represented as the differential. Since the action of D on $\mathfrak{g}[1] \rtimes \mathfrak{g}$ is by super Lie derivations, we can form the semi-direct product $(\mathfrak{g}[1] \rtimes \mathfrak{g}) \rtimes \mathbb{K}[-1]$. It is a graded Lie superalgebra, where the bracket relations between generators D, I_ξ, L_ξ are similar to the commutator relations between d, $\iota(\xi)$, $L(\xi)$, as listed above. Hence, a (graded, filtered) $\mathfrak{g}$-differential space may also be viewed as a representation of $(\mathfrak{g}[1] \rtimes \mathfrak{g}) \rtimes \mathbb{K}[-1]$, in the category of (graded, filtered) super spaces.

Remark 6.4. The condition $[L(\xi), \mathrm{d}] = 0$ says that each $L(\xi)$ is a cochain map. The condition $[\iota(\xi), \mathrm{d}] = L(\xi)$ shows that these cochain maps are all homotopic to 0, with $\iota(\xi)$ as homotopy operators. In particular, $L(\xi)$ induces the 0 action on cohomology.

Direct sums and tensor products of $\mathfrak{g}$-differential spaces are defined in an obvious way, making the category of $\mathfrak{g}$-differential spaces into a tensor category. We may hence consider algebra objects, coalgebra objects, Lie algebra objects, and so on. For instance, a *$\mathfrak{g}$-differential (Lie) algebra* is a $\mathfrak{g}$-differential space, which is also a super (Lie) algebra, such that $\mathrm{d}, L(\xi), \iota(\xi)$ are super (Lie) algebra derivations.

EXAMPLE 6.5. The differential Lie algebra $\mathfrak{g}[1] \rtimes \mathfrak{g}$ is an example of $\mathfrak{g}$-differential Lie algebra. (The action of $\mathfrak{g}[1] \rtimes \mathfrak{g}$ is just the adjoint representation.)

EXAMPLE 6.6. Suppose $\mathbb{K} = \mathbb{R}$, and let M is a manifold. An action of the Lie algebra $\mathfrak{g}$ on M is a Lie algebra homomorphism $\varrho\colon \mathfrak{g} \to \mathfrak{X}(M)$ into the Lie algebra of vector fields on M. Given such an action, the algebra $\mathcal{A} = \Omega(M)$ of differential forms becomes a $\mathfrak{g}$-differential algebra, with d the de Rham differential, and $\iota(\xi)$ and $L(\xi)$ the contractions and Lie derivatives with respect to the vector fields $\varrho(\xi)$.

Below we will encounter many other examples of $\mathfrak{g}$-differential algebras.

DEFINITION 6.7. Let E be a (graded, filtered) $\mathfrak{g}$-differential space. One defines the *basic subspace* E_{bas} to be the (graded, filtered) differential subspace, consisting of all $x \in E$ with $\iota(\xi)x = 0$ and $L(\xi)x = 0$ for all ξ. One calls

$$H_{\mathrm{bas}}(E) := H(E_{\mathrm{bas}}, \mathrm{d})$$

the *basic cohomology* of E.

Equivalently, E_{bas} is the subspace fixed by the action of $\mathfrak{g}[1] \rtimes \mathfrak{g}$. A morphism of $\mathfrak{g}$-differential spaces $E_1 \to E_2$ induces a morphism of differential spaces $(E_1)_{\mathrm{bas}} \to (E_2)_{\mathrm{bas}}$, hence of the basic cohomology.

We also define the *invariant subspace* $E_{\mathrm{inv}} = E^{\mathfrak{g}}$ to be the subspace annihilated by all $L(\xi)$, and the *horizontal subspace* E_{hor} to be the subspace annihilated by all $\iota(\xi)$. Thus $E_{\mathrm{bas}} = E_{\mathrm{hor}} \cap E^{\mathfrak{g}}$. Note however that the horizontal subspace is not a differential subspace, in general.

DEFINITION 6.8. A *connection* on a (graded, filtered) $\mathfrak{g}$-differential algebra $\mathcal{A}$ is an odd linear map (of degree 1, filtration degree 1)

$$\theta\colon \mathfrak{g}^* \to \mathcal{A}$$

with the properties,

(1) θ is $\mathfrak{g}$-equivariant: $\theta(L(\xi)\mu) = L(\xi)\theta(\mu)$,
(2) $\iota(\xi)\theta(\mu) = \langle \mu, \xi \rangle$.

A $\mathfrak{g}$-differential algebra admitting a connection is called *locally free.*

EXAMPLE 6.9. The $\mathfrak{g}$-differential algebra $\Omega(M)$ of differential forms on a $\mathfrak{g}$-manifold M is locally free if and only if the $\mathfrak{g}$ action on M is locally free, that is, $\xi \neq 0$ implies that the vector field $\varrho(\xi) \in \mathfrak{X}(M)$ has no zeroes.

The following is clear from the defnition:

PROPOSITION 6.10. *Let $\mathcal{A}$ be a (graded, filtered) $\mathfrak{g}$-differential algebra. If $\mathcal{A}$ is locally free, then the space of connections on $\mathcal{A}$ is an affine space with* $\mathrm{Hom}(\mathfrak{g}^*[-1], \mathcal{A}_{\mathrm{hor}})^{\mathfrak{g}}$ *as its space of motions.*

Here Hom denotes the space of morphisms of (graded, filtered) super spaces.

The *curvature* of a connection is an even map $\mathfrak{g}^* \to \mathcal{A}$ of degree 2, defined by the formula

$$F^\theta = \mathrm{d}\theta + \tfrac{1}{2}[\theta, \theta],$$

where we view θ as an element of $\mathcal{A} \otimes \mathfrak{g}$. Equivalently, if e_a is a basis of $\mathfrak{g}$, with dual basis e^a of $\mathfrak{g}^*$, and writing $\theta = \sum_a \theta^a e_a$, with $\theta^a \in \mathcal{A}$, we have $F^\theta = \sum_a (F^\theta)^a e_a$ with

$$(F^\theta)^a = \mathrm{d}\theta^a + \tfrac{1}{2} f^a_{bc} \theta^b \theta^c.$$

The curvature map $F^\theta \colon \mathfrak{g}^* \to \mathcal{A}$ is $\mathfrak{g}$-equivariant, and it takes values in $\mathcal{A}_{\mathrm{hor}}$. That is, it defines an element of $\mathrm{Hom}(\mathfrak{g}^*[-2], \mathcal{A}_{\mathrm{hor}})^{\mathfrak{g}}$.

7. The $\mathfrak{g}$-differential algebra $\wedge\mathfrak{g}^*$

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and $\wedge\mathfrak{g}^*$ the exterior algebra over the dual space. As explained in §2.1.2, the space of graded derivations is

$$\mathrm{Der}(\wedge\mathfrak{g}^*) \cong \mathrm{Hom}(\mathfrak{g}^*, \wedge\mathfrak{g}^*),$$

since any derivation is determined by its values on generators. In particular, $\mathrm{Der}^1(\wedge\mathfrak{g}^*) = \mathrm{Hom}(\mathfrak{g}^*, \wedge^2\mathfrak{g}^*)$. Let

$$\mathrm{d} \in \mathrm{Hom}(\mathfrak{g}^*, \wedge^2\mathfrak{g}^*)$$

be the map dual to the Lie bracket $[\cdot,\cdot] \colon \wedge^2 \mathfrak{g} \to \mathfrak{g}$. That is, for $\mu \in \mathfrak{g}^*$ and $\xi_1, \xi_2 \in \mathfrak{g}$,

$$\iota(\xi_1)\iota(\xi_2)\mathrm{d}\mu = \langle \mu, [\xi_1, \xi_2] \rangle. \tag{68}$$

Then $(\wedge\mathfrak{g}^*, \mathrm{d})$ is a graded differential algebra. To see that $\mathrm{d}^2 = 0$, it suffices to check on generators. It is actually interesting to consider the derivation d defined by an arbitrary linear map $[\cdot,\cdot] \colon \wedge^2 \mathfrak{g} \to \mathfrak{g}$ (not necessarily a Lie bracket). Define the *Jacobiator* $\mathrm{Jac} \colon \mathfrak{g}^* \to \wedge^3\mathfrak{g}^*$ by

$$\iota(\xi_1)\iota(\xi_2)\iota(\xi_3)\,\mathrm{Jac}(\mu) = \langle \mu, [\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]] \rangle,$$

so that $[\cdot,\cdot]$ is a Lie bracket if and only if $\mathrm{Jac} = 0$. Extend to a derivation $\mathrm{Jac} \in \mathrm{Der}^2(\wedge\mathfrak{g}^*)$

Proposition 7.1. *Let $\mathfrak{g}$ be a finite-dimensional vector space, and let $d \in \mathrm{Der}(\wedge\mathfrak{g}^*)$ be defined by duality to a linear map (bracket) $[\cdot,\cdot] \colon \wedge^2 \mathfrak{g} \to \mathfrak{g}$. Then d is a differential if and only if $[\cdot,\cdot]$ is a Lie bracket. In fact,*

$$d^2 = \mathrm{Jac}$$

where Jac *is the Jacobiator of the bracket.*

Proof. Define $L(\xi) \in \mathrm{Der}^0(\mathfrak{g})$ by $L(\xi) = [\mathrm{d}, \iota(\xi)]$. Thus $\langle L(\xi)\mu, \zeta \rangle = -\langle \mu, [\xi, \zeta] \rangle$ for all $\mu \in \mathfrak{g}^*, \xi, \zeta \in \mathfrak{g}$. Checking on generators, we find

$$[L(\xi), \iota(\zeta)] = \iota([\xi, \zeta]).$$

The square of d is a derivation, since it may be written $\mathrm{d}^2 = \frac{1}{2}[\mathrm{d},\mathrm{d}]$. To find what it is, we compute

$$\begin{aligned}\iota(\xi_1)\iota(\xi_2)\mathrm{d}\mathrm{d}\mu &= \iota(\xi_1)(L(\xi_2) - \mathrm{d}\iota(\xi_2))\mathrm{d}\mu \\ &= \big(\iota([\xi_1,\xi_2])\mathrm{d} + L(\xi_2)\iota(\xi_1)\mathrm{d} - \iota(\xi_1)\mathrm{d}L(\xi_2)\big)\mu \\ &= \big(L([\xi_1,\xi_2]) + L(\xi_2)L(\xi_1) - L(\xi_1)L(\xi_2)\big)\mu.\end{aligned}$$

Hence

$$\begin{aligned}\iota(\xi_1)\iota(\xi_2)\iota(\xi_3)\mathrm{d}\mathrm{d}\mu &= \iota(\xi_3)\iota(\xi_1)\iota(\xi_2)\mathrm{d}\mathrm{d}\mu \\ &= \langle\mu, [\xi_3,[\xi_1,\xi_2]] + [[\xi_3,\xi_2],\xi_1] - [[\xi_3,\xi_1],\xi_2]\rangle \\ &= \iota(\xi_1)\iota(\xi_2)\iota(\xi_3)\,\mathrm{Jac}(\mu).\end{aligned}$$

□

Suppose for the remainder of this Section that $\mathfrak{g}$ is a Lie algebra, so that $\mathrm{Jac} = 0$. The cohomology algebra $H(\wedge\mathfrak{g}^*, \mathrm{d})$ is called the *Lie algebra cohomology* of $\mathfrak{g}$, and is denoted $H(\mathfrak{g})$. We will frequently denote the differential on $\wedge\mathfrak{g}^*$ by $\mathrm{d}_\wedge$. Using dual bases $e_a \in \mathfrak{g}$ and $e^a \in \mathfrak{g}^*$, the Lie algebra differential may be written

$$\mathrm{d}_\wedge = \tfrac{1}{2}\sum_a e^a \circ L(e_a), \tag{69}$$

(with e^a acts by exterior multiplication), as is checked on generators. In particular, the $\mathfrak{g}$-invariants $(\wedge\mathfrak{g}^*)^{\mathfrak{g}}$ are all cocycles, defining a morphism of graded algebras

$$(\wedge\mathfrak{g}^*)^{\mathfrak{g}} \to H(\mathfrak{g}).$$

For $\mathfrak{g}$ reductive, this map is known to be an isomorphism. (See e.g. [?]. If $\mathbb{K} = \mathbb{R}$, and $\mathfrak{g}$ is the Lie algebra of a compact, connected Lie group, it follows by a simple averaging argument that the inclusion $(\wedge\mathfrak{g}^*)^{\mathfrak{g}} = (\wedge\mathfrak{g}^*)^G \hookrightarrow \wedge\mathfrak{g}^*$ induces an isomorphism in cohomology.)

Remark 7.2. Suppose $\mathbb{K} = \mathbb{R}$, and $\mathfrak{g}$ is the Lie algebra of a Lie group G. The identification of $\mathfrak{g}$ with left-invariant vector fields dualizes to an identification of $\mathfrak{g}^*$ with left-invariant 1-forms. This extends to an isomorphism of $\wedge\mathfrak{g}^*$ with the algebra of left-invariant differential forms on G, and the Chevalley-Eilenberg differential corresponds to the de Rham differential under this identification. If G is connected, then $(\wedge\mathfrak{g}^*)^{\mathfrak{g}} = (\wedge\mathfrak{g}^*)^G$ is identified with the bi-invariant forms on G.

Remark 7.3. Let $f^a_{bc} = \langle e^a, [e_b, e_c]\rangle$ be the structure constants of $\mathfrak{g}$ relative to the given basis. Then the differential on $\wedge\mathfrak{g}^*$ reads,

$$\mathrm{d}_\wedge = -\tfrac{1}{2} f^a_{bc} e^b e^c \iota(e_a).$$

To summarize, the exterior algebra $\wedge\mathfrak{g}^*$, with the Lie algebra differential $\mathrm{d}_\wedge$, and with the usual Lie derivatives and contraction operators, is a graded $\mathfrak{g}$-differential algebra. The horizontal and basic subspace of $\wedge\mathfrak{g}^*$ consists of the scalars, hence the basic cohomology is just $\mathbb{K}$. The map $\theta\colon \mathfrak{g}^* \to \wedge\mathfrak{g}^*$

given as the inclusion of $\wedge^1\mathfrak{g}^*$, is the unique connection on $\wedge\mathfrak{g}^*$; the curvature is zero (since $(\wedge^2\mathfrak{g}^*)_{\mathrm{hor}} = 0$).

More generally, suppose $L_V\colon \mathfrak{g} \to \mathrm{End}(V)$ is a $\mathfrak{g}$-representation on a vector space V. Let

$$C^\bullet(\mathfrak{g}, V) = V \otimes \wedge^\bullet\mathfrak{g}^*$$

with grading induced from the grading on the exterior algebra, and define the (Chevalley-Eilenberg) differential

$$\mathrm{d}_{CE} = \sum_a L_V(e_a) \otimes e^a + 1 \otimes \mathrm{d}_\wedge \tag{70}$$

Then $(C^\bullet(\mathfrak{g}, V), \mathrm{d}_{CE})$ is a graded differential space. The cohomology groups of the complex $C^\bullet(\mathfrak{g}, V)$ are denoted $H^\bullet(\mathfrak{g}, V)$. Note that

$$H^0(\mathfrak{g}, V) = V^{\mathfrak{g}}.$$

With the Lie derivatives $L(\xi) = L_V(\xi) \otimes 1 + 1 \otimes L_\wedge(\xi)$ and contractions $\iota(\xi) = 1 \otimes \iota_\wedge(\xi)$, the complex $C^\bullet(\mathfrak{g}, V)$ becomes a graded $\mathfrak{g}$-differential space. The basic subcomplex is $V^{\mathfrak{g}}$ with the zero differential.

8. $\mathfrak{g}$-homotopies

We next generalize the definition of homotopies:

Definition 8.1. Let $\phi_0, \phi_1\colon E \to E'$ be morphisms of (graded, filtered) $\mathfrak{g}$-differential spaces.

(1) A $\mathfrak{g}$-*homotopy* between ϕ_0, ϕ_1 of is a morphism of (graded, filtered) $\mathfrak{g}$-differential spaces $\phi\colon E \to \mathbb{K}[t, \mathrm{d}t] \otimes E'$ with

$$\phi_0 = (\pi_0 \otimes 1) \circ \phi, \;\; \phi_1 = (\pi_1 \otimes 1) \circ \phi.$$

(2) A $\mathfrak{g}$-*homotopy operator* between ϕ_0, ϕ_1 is morphism of (graded, filtered) super spaces $h\colon E[1] \to E'$ such that

$$h \circ \iota(\xi) + \iota'(\xi) \circ h = 0,$$

$$h \circ \mathrm{d} + \mathrm{d}' \circ h = \phi_1 - \phi_0.$$

Thus h is odd (of degree -1 in the graded or filtered cases). The definition of h implies that it intertwines Lie derivatives as well:

$$\begin{aligned}
[h, L(\xi)] &= [h, [\mathrm{d}, \iota(\xi)]] \\
&= [[h, \mathrm{d}], \iota(\xi)] - [\mathrm{d}, [h, \iota(\xi)]] \\
&= [\phi_1 - \phi_0, \iota(\xi)] = 0.
\end{aligned}$$

(By a small abuse of notation, we wrote $[h, L(\xi)]$ for $h \circ L(\xi) - L'(\xi) \circ h$, etc.) In other words, a $\mathfrak{g}$-homotopy operator h is a homotopy operator such that the map $h\colon E[1] \to E'$ is $\mathfrak{g}[1] \rtimes \mathfrak{g}$-equivariant.

The discussion of homotopies in Section §6.3 extends to the case of $\mathfrak{g}$-homotopies, with the obvious changes. In particular, the same argument as for Proposition §6 3.5 shows that ϕ_0, ϕ_1 are $\mathfrak{g}$-homotopic if and only if there is a $\mathfrak{g}$-homotopy operator. In this case, the maps in basic cohomology are the same:

$$H_{\mathrm{bas}}(\phi_0) = H_{\mathrm{bas}}(\phi_1)\colon H_{\mathrm{bas}}(E) \to H_{\mathrm{bas}}(E').$$

9. The Weil algebra

Consider the Koszul algebra for $\mathfrak{g}^*[-1]$,

$$S(E_{\mathfrak{g}^*}[-1]) = S(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*).$$

As before, we associate to each $\mu \in \mathfrak{g}^*$ the degree 1 generators $\mu \in \mathfrak{g}^*[-1] = \wedge^1\mathfrak{g}^*$ and the degree 2 generators $\overline{\mu} \in \mathfrak{g}^*[-2] = S^1\mathfrak{g}^*$, so that $\mathrm{d}\mu = \overline{\mu}$, $\mathrm{d}\overline{\mu} = 0$. The coadjoint $\mathfrak{g}$-representation on $\mathfrak{g}^*$ defines a representation on $E_{\mathfrak{g}^*}$, commuting with the differentia. The resulting $\mathfrak{g}$-representation by derivations of $S(E_{\mathfrak{g}^*}[-1])$ is the tensor product of the co-adjoint representations on $S(\mathfrak{g}^*)$ and $\wedge(\mathfrak{g}^*)$; the generators for the action will be denoted $L(\xi)$. To turn $S(E_{\mathfrak{g}^*}[-1])$ into a $\mathfrak{g}$-differential algebra, we need to define the contraction operators. The action of $\iota(\xi)$ on $\overline{\mu}$ is determined:

$$\iota(\xi)\overline{\mu} = \iota(\xi)\mathrm{d}\mu = L(\xi)\mu - \mathrm{d}\iota(\xi)\mu = L(\xi)\mu.$$

On the degree 1 generators, it is natural to take $\iota(\xi)\mu = \langle\mu, \xi\rangle$. It is straightforward to check the relations involving $\iota(\xi)$ on generators, so that we have turned $S(E_{\mathfrak{g}^*}[-1])$ into a $\mathbb{Z}$-graded $\mathfrak{g}$-differential algebra. [2]

Definition 9.1. The graded $\mathfrak{g}$-differential algebra

$$W\mathfrak{g} = S(E_{\mathfrak{g}^*}[-1])$$

is called the *Weil algebra* for $\mathfrak{g}$.

As a special case of Corollary §6.4.5 (and the subsequent Remark) we have,

Proposition 9.2. *The Weil algebra $W\mathfrak{g}$, as well as its $\mathfrak{g}$-invariant part $(W\mathfrak{g})^{\mathfrak{g}}$, are acyclic differential algebras.*

Proposition 9.3. *The Weil algebra is locally free, with connection*

$$\theta_W\colon \mathfrak{g}^* \to W\mathfrak{g},\ \ \mu \mapsto \mu.$$

The curvature of the connection on $W\mathfrak{g}$ is given by

$$F^{\theta_W}\colon \mathfrak{g}^* \to W^2\mathfrak{g},\ \ \mu \mapsto \overline{\mu} - d_\wedge(\mu).$$

Proof. It is immediate that $\theta_W(\mu) = \mu$ is a connection.

In terms of a basis of $\mathfrak{g}$, the components of the connection 1-form on $W\mathfrak{g}$ are $\theta^a_W = e^a$; hence those of the curvature form are

$$(F^{\theta_W})^a = \mathrm{d}e^a + \tfrac{1}{2}\sum_{bc} f^a_{bc} e^b e^c = \mathrm{d}e^a - \mathrm{d}_\wedge e^a.$$

∎

[2] More generally, for any $\kappa \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})^{\mathfrak{g}}$, one obtains a $\mathfrak{g}$-differential algebra by putting $\iota(\xi)\mu = \langle\mu, \kappa(\xi)\rangle$.

REMARK 9.4. We may verify directly that $\overline{\mu} - \mathrm{d}_\wedge(\mu)$ is horizontal:

$$\iota(\xi)\overline{\mu} - \mathrm{d}_\wedge(\mu) = L(\xi)\mu - L_\wedge(\mu) + \mathrm{d}_\wedge(\iota(\xi)\mu) = 0,$$

since $L(\xi)$ agrees with $L_\wedge(\xi)$ on $\wedge\mathfrak{g}^* \subset W\mathfrak{g}$, and since $\mathrm{d}_\wedge$ vanishes on scalars. (Thus $\overline{\mu} - \mathrm{d}_\wedge\mu$ is the horizontal projection of $\overline{\mu}$ – this gives an alternative proof of the curvature formula.)

By definition, the Weil algebra $W\mathfrak{g}$ is a tensor product $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$ where $S\mathfrak{g}^*$ is the symmetric algebra generated by the variables $\overline{\mu}$ and $\wedge\mathfrak{g}^*$ is the exterior algebra generated by the variables μ. It is often convenient however, to replace the generators $\mu, \overline{\mu}$ with generators $\mu, \widehat{\mu}$, where

$$\widehat{\mu} := \overline{\mu} - \mathrm{d}_\wedge\mu \in W^2\mathfrak{g}.$$

are the curvature variables. Thus, we obtain a second identification of $W\mathfrak{g}$ with $S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$, where now $S\mathfrak{g}^*$ is the symmetric algebra generated by the variables $\widehat{\mu}$.

The main advantage of this change of variables is that the contraction operators simplify to

$$\iota(\xi)\mu = \langle\mu, \xi\rangle, \quad \iota(\xi)\widehat{\mu} = 0.$$

THEOREM 9.5. *The horizontal and basic subspaces of the Weil algebra $W\mathfrak{g}$ are*

$$(W\mathfrak{g})_{\mathrm{hor}} = S\mathfrak{g}^*, \quad (W\mathfrak{g})_{\mathrm{bas}} = (S\mathfrak{g}^*)_{\mathfrak{g}},$$

where $S\mathfrak{g}^*$ is the symmetric algebra generated by the variables $\widehat{\mu}$. The differential on the basic subcomplex is just 0*, so*

$$H_{\mathrm{bas}}(W\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}.$$

PROOF. The description of the horizontal and basic subspaces is immediate. Since the basic subcomplex $(W\mathfrak{g})_{\mathrm{bas}}$ only contains elements of even degree, its differential is zero, hence the complex coincides with its cohomology. □

We should still describe the differential of $W\mathfrak{g}$ in the new variables.

THEOREM 9.6. *Identify $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$, where $\wedge\mathfrak{g}^*$ is the exterior algebra generated by the variables μ, and $S\mathfrak{g}^*$ is the symmetric algebra generated by the variables $\widehat{\mu}$. Let d_{CE} be the Chevalley-Eilenberg differential* (70) *for the $\mathfrak{g}$-module $S\mathfrak{g}^*$, and d_K the Koszul differential relative to the generators $\mu, \widehat{\mu}$, that is, $d_K\mu = \widehat{\mu}$, $d_K\widehat{\mu} = 0$. Then the super derivations d_{CE}, d_K commute, and the Weil differential is their sum:*

$$d = d_K + d_{CE}.$$

PROOF. For all $\mu \in \mathfrak{g}^*$, we have

$$\mathrm{d}\mu = \overline{\mu} = \widehat{\mu} + \mathrm{d}_\wedge(\mu) = \mathrm{d}_K\mu + \mathrm{d}_{CE}\mu$$

$$\mathrm{d}\widehat{\mu} = -\mathrm{d}(\mathrm{d}_\wedge\mu) = \sum_i \widehat{L(e_i)\mu} \otimes e^i = \mathrm{d}_{CE}\widehat{\mu} = (\mathrm{d}_K + \mathrm{d}_{CE})\widehat{\mu}.$$

This gives the equality of derivations $\mathrm{d} = \mathrm{d}_K + \mathrm{d}_{CE}$ on generators, and hence everywhere. The fact that $\mathrm{d}_K, \mathrm{d}_{CE}$ commute follows since $0 = \mathrm{d}^2 - \mathrm{d}_K^2 - \mathrm{d}_{CE}^2 = \mathrm{d}_K \mathrm{d}_{CE} + \mathrm{d}_{CE} \mathrm{d}_K$. $\square$

REMARK 9.7. The automorphism of $S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$, given on generators by $\mu \mapsto \mu$, $\widehat{\mu} \mapsto \frac{1}{r}\widehat{\mu}$, for $r \neq 0$, intertwines Lie derivatives and contractions, and takes d to the differential

$$\mathrm{d}^{(r)} = r\mathrm{d}_K + \mathrm{d}_{CE}.$$

Let $(W\mathfrak{g})^{(r)} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$ be the family of $\mathfrak{g}$-differential algebras with the derivations $\mathrm{d}^{(r)}, \iota(\xi), L(\xi)$. For $r \neq 0$ these are all isomorphic, but for $r = 0$ the family degenerates to the Chevalley-Eilenberg complex for the $\mathfrak{g}$-module $S\mathfrak{g}^*$. Later, we will find it convenient to work with $r = 2$.

10. Chern-Weil homomorphisms

The Weil algebra is universal among commutative $\mathfrak{g}$-differential algebras with connection.

PROPOSITION 10.1 (Universal property of the Weil algebra). *For any commutative (graded, filtered) $\mathfrak{g}$-differential algebra $\mathcal{A}$ with connection $\theta_{\mathcal{A}}$, there is a unique morphism of (graded, filtered) $\mathfrak{g}$-differential algebras*

$$c\colon W\mathfrak{g} \to \mathcal{A} \tag{71}$$

such that $c \circ \theta_W = \theta_{\mathcal{A}}$.

PROOF. Suppose $\mathcal{A}$ is a commutative $\mathfrak{g}$-differential algebra with connection. By the universal property of Koszul algebras, the map $\theta\colon \mathfrak{g}^* \to \mathcal{A}^{\bar{1}}$ extends uniquely to a homomorphism of differential algebras $c\colon W\mathfrak{g} \to \mathcal{A}$. The calculation

$$\begin{aligned} \iota(\xi)c(\overline{\mu}) &= \iota(\xi)\mathrm{d}\theta(\mu) \\ &= L(\xi)\theta(\mu) - \mathrm{d}\iota(\xi)\theta(\mu) \\ &= \theta(L(\xi)\mu) - \mathrm{d}\langle \mu, \xi \rangle \\ &= c(L(\xi)\mu) = c(\iota(\xi)\overline{\mu}), \end{aligned}$$

together with $\iota(\xi)c(\mu) = \iota(\xi)\theta(\mu) = \langle \mu, \xi \rangle = c(\iota(\xi)\mu)$, shows that c intertwines contractions. Since $L(\xi) = [\iota(\xi), \mathrm{d}]$, it intertwines Lie derivatives as well. $\square$

The homomorphism c is called the *characteristic homomorphism* for the connection θ.

EXAMPLES 10.2. (1) The connection $\theta_\wedge$ on the commutative $\mathfrak{g}$-differential algebra $\mathcal{A} = \wedge\mathfrak{g}^*$ defines a morphism of $\mathfrak{g}$-differential algebras

$$W\mathfrak{g} \to \wedge\mathfrak{g}^*. \tag{72}$$

This is the map taking the generators $\widehat{\mu}$ to zero.

(2) For any commutative $\mathfrak{g}$-differential algebra $\mathcal{A}$, the tensor product $W\mathfrak{g} \otimes \mathcal{A}$ carries a connection $\theta_W \otimes 1$. This defines a morphism $W\mathfrak{g} \to W\mathfrak{g} \otimes \mathcal{A}$, $w \mapsto w \otimes 1$, and hence a map in basic cohomology
$$(S\mathfrak{g}^*)^{\mathfrak{g}} = H_{\mathrm{bas}}(W\mathfrak{g}) \to H_{\mathrm{bas}}(W\mathfrak{g} \otimes \mathcal{A}).$$

The argument from the proof of Theorem 4.4 extends to the case of $\mathfrak{g}$-differential algebras.

THEOREM 10.3. *If $\mathcal{A}$ is a commutative (graded, filtered) $\mathfrak{g}$-differential algebra, any two morphisms of (graded, filtered) $\mathfrak{g}$-differential algebras $c_0, c_1 \colon W\mathfrak{g} \to \mathcal{A}$ are $\mathfrak{g}$-homotopic.*

PROOF. Let $\theta_0, \theta_1 \colon \mathfrak{g}^* \to \mathcal{A}$ be the connections defined by the restrictions of c_0, c_1. Then
$$\theta = (1-t)\theta_0 + t\theta_1$$
is a connection on $\mathbb{K}[t, \mathrm{d}t] \otimes \mathcal{A}$, and its characteristic homomorphism
$$c \colon W\mathfrak{g} \to \mathbb{K}[t, \mathrm{d}t] \otimes \mathcal{A}$$
is the desired $\mathfrak{g}$-homotopy. $\square$

As an immediate consequence, if $\mathcal{A}$ is a commutative $\mathfrak{g}$-differential algebra admitting a connection θ, then the resulting algebra homomorphism in basic cohomology
$$(S\mathfrak{g}^*)^{\mathfrak{g}} = H_{\mathrm{bas}}(W\mathfrak{g}) \to H_{\mathrm{bas}}(\mathcal{A}) \tag{73}$$
is independent of the connection.

DEFINITION 10.4. let $\mathcal{A}$ be a $\mathfrak{g}$-differential algebra with connection. The resulting algebra homorphism
$$(S\mathfrak{g}^*)^{\mathfrak{g}} \to H_{\mathrm{bas}}(\mathcal{A})$$
is called the *Chern-Weil homomorphism.*

REMARK 10.5. This terminology goes back to the differential geometry of principal bundles. Let G be a Lie group, and $P \to B$ a principal G-bundle with G-equivariant connection $\theta \colon \mathfrak{g}^* \to \Omega^1(P)$. In this case, the basis subcomplex $\Omega(P)_{\mathrm{bas}}$ of G-invariant, $\mathfrak{g}$-horizontal elements is isomorphic to the de Rham complex $\Omega(B)$. The elements in $H_{\mathrm{deRham}}(B) = H(\Omega(B))$ obtained as images under the Chern-Weil homomorphism are called the *characteristic classes.*

PROPOSITION 10.6. *Let $\mathcal{A}$ be a commutative $\mathfrak{g}$-differential algebra with connection, and denote by $c \colon W\mathfrak{g} \to \mathcal{A}$ the characteristic homomorphism. Then the map*
$$\phi \colon W\mathfrak{g} \otimes \mathcal{A} \to \mathcal{A},\ w \otimes x \mapsto c(w)x$$
is a $\mathfrak{g}$-homotopy equivalence, with $\mathfrak{g}$-homotopy inverse the inclusion,
$$\psi \colon \mathcal{A} \to W\mathfrak{g} \otimes \mathcal{A},\ x \mapsto 1 \otimes x.$$

PROOF. Clearly, $\phi \circ \psi = \mathrm{id}_{\mathcal{A}}$. On the other hand, $(\psi \circ \phi)(w \otimes x) = 1 \otimes c(w)x$. Let $\tau_0, \tau_1 \colon W\mathfrak{g} \to W\mathfrak{g} \otimes \mathcal{A}$ be the characteristic homomorphisms for the connections $\theta_0 = 1 \otimes \theta_{\mathcal{A}}$ and $\theta_1 = \theta_W \otimes 1$ on $W\mathfrak{g} \otimes \mathcal{A}$. Thus $\tau_0(w) = 1 \otimes c(w)$, $\tau_1(w) = w \otimes 1$. We have

$$\psi \circ \phi = (\mathrm{id}_{W\mathfrak{g}} \otimes m_{\mathcal{A}}) \circ (\tau_0 \otimes \mathrm{id}_{\mathcal{A}}),$$
$$\mathrm{id}_{W\mathfrak{g}\otimes\mathcal{A}} = (\mathrm{id}_{W\mathfrak{g}} \otimes m_{\mathcal{A}}) \circ (\tau_1 \otimes \mathrm{id}_{\mathcal{A}}),$$

where $m_{\mathcal{A}}$ is the multiplication in $\mathcal{A}$. By Theorem 10.3, τ_0 is $\mathfrak{g}$-homotopic to τ_1. Since $\mathfrak{g}$-homotopies can be composed, it follows that $\psi \circ \phi$ is $\mathfrak{g}$-homotopic to $\mathrm{id}_{W\mathfrak{g}\otimes\mathcal{A}}$. □

11. The non-commutative Weil algebra $\tilde{W}\mathfrak{g}$

The Weil algebra $W\mathfrak{g} = S(E_{\mathfrak{g}^*}[-1])$ is characterized by its universal property among the commutative $\mathfrak{g}$-differential algebras with connection. To obtain a similar universal object among non-commutative $\mathfrak{g}$-differential algebras with connection, we only have to replace the super symmetric algebra with the tensor algebra. As a differential algebra, $\tilde{W}\mathfrak{g} = T(E_{\mathfrak{g}^*}[-1])$, is the non-commutative Koszul algebra (cf. §6.4), freely generated by degree 1 generators μ and degree 2 generators $\overline{\mu}$. The formulas for the contractions are given on generators by just the same formulas as for $W\mathfrak{g}$:

$$\iota(\xi)\mu = \langle \mu, \xi \rangle, \qquad \iota(\xi)\overline{\mu} = L(\xi)\mu.$$

DEFINITION 11.1. The graded $\mathfrak{g}$-differential algebra

$$\tilde{W}\mathfrak{g} = T(E_{\mathfrak{g}^*}[-1])$$

is called the *noncommutative Weil algebra*.[3]

Most of the results for the commutative Weil algebra carry over to the non-commutative case, with essentially the same proofs (simply replace W with $\tilde{W}$). We will not repeat the proofs, but just state the results.

(1) The non-commutative Weil algebra $\tilde{W}\mathfrak{g}$ is locally free, with connection
$$\theta_{\tilde{W}} \colon \mathfrak{g}^* \to \tilde{W}\mathfrak{g}, \ \mu \mapsto \mu.$$
(2) Given a (graded, filtered) $\mathfrak{g}$-differential algebra $\mathcal{A}$ with connection $\theta_{\mathcal{A}}$, there is a unique morphism of (graded, filtered) $\mathfrak{g}$-differential algebras
$$c \colon \tilde{W}\mathfrak{g} \to \mathcal{A}$$
such that $c \circ \theta_W = \theta_{\mathcal{A}}$. (Cf. Proposition 10.1).
(3) Given a (graded, filtered) $\mathfrak{g}$-differential algebra $\mathcal{A}$, any two morphisms of (graded, filtered) $\mathfrak{g}$-differential algebras $c_0, c_1 \colon \tilde{W}\mathfrak{g} \to \mathcal{A}$

[3]The algebra $\tilde{W}\mathfrak{g}$ is different from the noncommutative Weil algebra $\mathcal{W}\mathfrak{g}$ of [**4**], which we will discuss below under the name of *quantum Weil algebra*.

are $\mathfrak{g}$-homotopic. (cf. Theorem 10.3) We hence see that for a non-commutative $\mathfrak{g}$-differential algebra $\mathcal{A}$ with connection, algebra homomorphism

$$H_{\mathrm{bas}}(\tilde{W}\mathfrak{g}) \to H_{\mathrm{bas}}(\mathcal{A}),$$

induced by characteristic homomorphism $c\colon \tilde{W}\mathfrak{g} \to \mathcal{A}$ does not depend on the connection. Theorem 11.3 below shows that $H_{\mathrm{bas}}(\tilde{W}\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}$; hence we have generalized the Chern-Weil homomorphism to the non-commutative setting.

(4) Let $\mathcal{A}$ be a $\mathfrak{g}$-differential algebra with connection, and let $c\colon \tilde{W}\mathfrak{g} \to \mathcal{A}$ be the characteristic homomorphism. Then the map

$$\phi\colon \tilde{W}\mathfrak{g} \otimes \mathcal{A} \to \mathcal{A},\ \ w \otimes x \mapsto c(w)x$$

is a $\mathfrak{g}$-homotopy equivalence, with $\mathfrak{g}$-homotopy inverse the inclusion, $\psi\colon \mathcal{A} \to \tilde{W}\mathfrak{g} \otimes \mathcal{A},\ \ x \mapsto 1 \otimes x$. (cf. Proposition 10.6).

In the commutative case, we found that the horizontal subalgebra of $W\mathfrak{g}$ is the symmetric algebra generated by the curvature variables $\hat{\mu}$. As a consequence we could read off that $H_{\mathrm{bas}}(W\mathfrak{g}) = (W\mathfrak{g})_{\mathrm{bas}} = (S\mathfrak{g}^*)^{\mathfrak{g}}$. This aspect of $W\mathfrak{g}$ does *not* carry over to $\tilde{W}\mathfrak{g}$, and changing the variables to $\mu, \hat{\mu}$ does not appear useful. We will show however that the quotient map

$$\tilde{W}\mathfrak{g} \to W\mathfrak{g}$$

(the characteristic homomorphism for $W\mathfrak{g}$, regarded as a non-commutative $\mathfrak{g}$-differential algebra) is a $\mathfrak{g}$-homotopy equivalence. In particular, this will show $H_{\mathrm{bas}}(\tilde{W}\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}$.

Proposition 11.2. *The map*

$$W\mathfrak{g} = S(E_{\mathfrak{g}^*}[-1]) \to \tilde{W}\mathfrak{g} = T(E_{\mathfrak{g}^*}[-1]) \tag{74}$$

given by the inclusion of symmetric tensors (i.e. the extension of $E_{\mathfrak{g}^*}[-1] \hookrightarrow T(E_{\mathfrak{g}^*}[-1])$ by symmetrization) is a morphism of $\mathfrak{g}$-differential spaces.

Proof. The proof is similar to that of Proposition §6.5.2. However, in this case Lemma §6.5.1 does not immediately apply, since the differential space $E_{\mathfrak{g}^*}[-1]$ is not a $\mathfrak{g}$-differential space. Instead, consider the graded vector space

$$E_{\mathfrak{g}^*}[-1] \oplus \mathbb{K}\mathsf{c}$$

where c is a generator of degree 0. It carries the structure of a graded $\mathfrak{g}$-differential space, with the given differential and Lie derivative on $E_{\mathfrak{g}^*}[-1]$, with $\iota(\xi)\mu = \langle \mu, \xi \rangle \mathsf{c}$, $\iota(\xi)\overline{\mu} = L(\xi)\mu$, and with trivial action of $\iota(\xi), L(\xi), \mathrm{d}$ on c. We have,

$$W\mathfrak{g} = S(E_{\mathfrak{g}^*}[-1] \oplus \mathbb{K}\mathsf{c}) / < \mathsf{c} - 1 > . \tag{75}$$

Let $E_{\mathfrak{g}^*}[-1] \oplus \mathbb{K}\mathsf{c} \to \tilde{W}\mathfrak{g}$ be the map given by the inclusion of $E_{\mathfrak{g}^*}[-1]$ on the first summand, and by the map $\mathsf{c} \mapsto 1$ on the second summand. This map is a morphism of $\mathfrak{g}$-differential spaces; hence Lemma 5.1 shows that it extends to a map of $\mathfrak{g}$-differential spaces

$$S(E_{\mathfrak{g}^*}[-1] \oplus \mathbb{K}\mathsf{c}) \to \tilde{W}\mathfrak{g}.$$

The ideal in $S(E_{\mathfrak{g}^*}[-1] \oplus \mathbb{K}\mathsf{c})$ generated by $\mathsf{c} - 1$ is a $\mathfrak{g}$-differential subspace, contained in the kernel of this map. Since 'symmetrizing' and 'setting c equal to 1' commute, this is the same as the map (74). □

THEOREM 11.3. *The quotient map $\phi\colon \tilde{W}\mathfrak{g} \to W\mathfrak{g}$ is a $\mathfrak{g}$-homotopy equivalence, with homotopy inverse $\psi\colon W\mathfrak{g} \to \tilde{W}\mathfrak{g}$ given by symmetrization, $S(E_{\mathfrak{g}^*}[-1]) \to T(E_{\mathfrak{g}^*}[-1])$.*

PROOF. The proof is parallel to that of Proposition 5.3. The main point is that the two morphisms $c_0, c_1\colon \tilde{W}\mathfrak{g} \to \tilde{W}\mathfrak{g} \otimes W\mathfrak{g}$ given by

$$c_0(w) = w \otimes 1,\ c_1(w) = 1 \otimes \phi(w)$$

are the characteristic homomorphisms for the two natural connections on $\tilde{W}\mathfrak{g} \otimes W\mathfrak{g}$. Hence they are $\mathfrak{g}$-homotopic (Property (3) above), and so is their composition with

$$\tilde{W}\mathfrak{g} \otimes W\mathfrak{g} \to \tilde{W}\mathfrak{g},\ w \otimes w' \mapsto w\,\mathrm{sym}(w')$$ □

COROLLARY 11.4. *The quotient map $\tilde{W}\mathfrak{g} \to W\mathfrak{g}$ induces an isomorphism $H_{\mathrm{bas}}(\tilde{W}\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}$.*

12. Equivariant cohomology of $\mathfrak{g}$-differential spaces

DEFINITION 12.1. The *equivariant cohomology* of a $\mathfrak{g}$-differential space E is the cohomology of the basic subcomplex of $W\mathfrak{g} \otimes E$:

$$H_{\mathfrak{g}}(E) = H_{\mathrm{bas}}(W\mathfrak{g} \otimes E). \tag{76}$$

The left multiplication of $(W\mathfrak{g})_{\mathrm{bas}} = (S\mathfrak{g}^*)^{\mathfrak{g}}$ on $(W\mathfrak{g} \otimes E)_{\mathrm{bas}}$ gives $H_{\mathfrak{g}}(E)$ the structure of a module over $(S\mathfrak{g}^*)^{\mathfrak{g}}$.

If $\mathcal{A}$ is a $\mathfrak{g}$-differential algebra, then $W\mathfrak{g} \otimes \mathcal{A}$ is a $\mathfrak{g}$-differential algebra, hence $H_{\mathfrak{g}}(\mathcal{A})$ inherits a super algebra structure. Similarly, if a $\mathfrak{g}$-differential space E is a module over the $\mathfrak{g}$-differential algebra $\mathcal{A}$, then $H_{\mathfrak{g}}(E)$ becomes a module over $H_{\mathfrak{g}}(\mathcal{A})$.

REMARK 12.2. By Theorem 11.3, one can replace $W\mathfrak{g}$ with $\tilde{W}\mathfrak{g}$ in the definition of $H_{\mathfrak{g}}(E)$. If $\mathcal{A}$ is a $\mathfrak{g}$-differential algebra, the resulting product on $H_{\mathfrak{g}}(\mathcal{A})$ does not depend on the use of $W\mathfrak{g}$ or $\mathcal{W}\mathfrak{g}$ in its definition. Indeed, since the quotient map $\tilde{W}\mathfrak{g} \otimes \mathcal{A} \to W\mathfrak{g} \otimes \mathcal{A}$ is a super algebra morphism, the induced isomorphism in basic cohomology is a super algebra morphism.

EXAMPLE 12.3. If $\mathfrak{g}$ is a real Lie algebra, and M is a $\mathfrak{g}$-manifold, the cohomology group $H_{\mathfrak{g}}(M) = H_{\mathfrak{g}}(\Omega(M))$ is called the *equivariant de Rham cohomology* of M. Suppose $\mathfrak{g}$ is a Lie algebra of a compact Lie group G, that M is compact, and that the action of $\mathfrak{g}$ integrates to an action of G on M. According to a Theorem of H. Cartan [**19**], $H_{\mathfrak{g}}(M)$ coincides in this case with the equivariant cohomology $H_G(M,\mathbb{R}) = H(EG \times_G M, \mathbb{R})$. Thus, $W\mathfrak{g}$, $\tilde{W}\mathfrak{g}$ may be thought of as algebraic counterparts to the classifying bundle EG.

PROPOSITION 12.4. *If $\mathcal{A}$ is a $\mathfrak{g}$-differential algebra admitting a connection, then its equivariant cohomology (cf.* (76)*) is canonically isomorphic to its basic cohomology: Thus*

$$H_{\mathfrak{g}}(\mathcal{A}) \cong H_{\mathrm{bas}}(\mathcal{A})$$

as super algebras.

PROOF. The $\mathfrak{g}$-homotopy equivalence $\mathcal{A} \to \tilde{W}\mathfrak{g} \otimes \mathcal{A}$, $x \mapsto 1 \otimes x$ (cf. §6.11 (4)) restricts to a homotopy equivalence of the basic subcomplexes, hence to an isomorphism $H_{\mathrm{bas}}(\mathcal{A}) \to H_{\mathfrak{g}}(\mathcal{A})$. $\square$

This result admits a generalization to (graded, filtered) $\mathfrak{g}$-differential spaces (E, d) having the structure of a module over $\tilde{W}\mathfrak{g}$.[4]

DEFINITION 12.5. A module over a (graded, filtered) $\mathfrak{g}$-differential algebra $\mathcal{A}$ is a (graded, filtered) $\mathfrak{g}$-differential space E, which is also a module over $\mathcal{A}$, in such a way that the module action $\phi\colon \mathcal{A} \otimes E \to E$ is a morphism of (graded, filtered) $\mathfrak{g}$-differential spaces.

Note that if $\mathcal{A}$ is a $\mathfrak{g}$-differential algebra with connection θ, then the characteristic homomorphism $c\colon \tilde{W}\mathfrak{g} \to \mathcal{A}$ makes $\mathcal{A}$ into a module over $\tilde{W}\mathfrak{g}$, via $\phi(w \otimes x) = c(w)x$. If the super algebra $\mathcal{A}$ is commutative, then $\mathcal{A}$ becomes a module over $W\mathfrak{g}$.

Let us make the convention that in the category of super spaces, $\mathrm{End}(E)$ denotes the super algebra of *all* linear maps $E \to E$, but in the cateory of graded (resp. filtered) super spaces, we take $\mathrm{End}(E)$ to be with the algebra of linear maps $E \to E$ of finite degree (resp. finite filtration degree). With this convention, if E is a (graded, filtered) $\mathfrak{g}$-differential space, the space $\mathrm{End}(E)$ becomes a (graded, filtered) $\mathfrak{g}$-differential algebra. The condition for a module E over a (graded, filtered) $\mathfrak{g}$-differential algebra $\mathcal{A}$ is equivalent to the condition that the map $\mathcal{A} \to \mathrm{End}(E)$ given by the module action is a morphism of (graded, filtered) $\mathfrak{g}$-differential algebras.

PROPOSITION 12.6. *Any two (graded, filtered) module structures*

$$\phi_0, \phi_1\colon \tilde{W}\mathfrak{g} \otimes E \to E$$

over the $\mathfrak{g}$-differential algebra $\tilde{W}\mathfrak{g}$ are $\mathfrak{g}$-homotopic.

PROOF. The corresponding morphisms of $\mathfrak{g}$-differential algebras $\tilde{W}\mathfrak{g} \to \mathrm{End}(E)$ are $\mathfrak{g}$-homotopic, by §6.11 (3). $\square$

PROPOSITION 12.7. *Let E be a (graded, filtered) module over the (graded, filtered) $\mathfrak{g}$-differential algebra $\tilde{W}\mathfrak{g}$. Then the module map is a $\mathfrak{g}$-homotopy equivalence, with homotopy inverse the inclusion $E \to \tilde{W}\mathfrak{g} \otimes E$, $v \mapsto 1 \otimes v$.*

[4] The following results will not be needed elsewhere in this book.

PROOF. The tensor product $E' = \tilde{W}\mathfrak{g} \otimes E$ is a $\tilde{W}\mathfrak{g}$-module in two ways, with $w \in \tilde{W}\mathfrak{g}$ acting on $w_1 \otimes v$ by $ww_1 \otimes v$ or by $(-1)^{|w||w_1|}w_1 \otimes wv$. By the previous Proposition, these two module structures are $\mathfrak{g}$-homotopic. Now argue as in the proof of Proposition 10.6. $\square$

As a consequence, the map on basic subcomplexes $(\tilde{W}\mathfrak{g} \otimes E)_{\mathrm{bas}} \to E_{\mathrm{bas}}$ induces an isomorphism in cohomology,

$$H_{\mathfrak{g}}(E) \cong H_{\mathrm{bas}}(E).$$

The book of Guillemin-Sternberg [**33**] gives a detailed discussion of modules over the *commutative* Weil algebra $W\mathfrak{g}$ (under the name '$W^\star$-modules').

13. Transgression in the Weil algebra

We will briefly discuss *transgression* in the Weil algebra $W\mathfrak{g}$. Recall that $W\mathfrak{g}$, and also its invariant part $(W\mathfrak{g})^{\mathfrak{g}}$, are acyclic differential algebras. That is, the augmentation map $\epsilon\colon (W\mathfrak{g})^{\mathfrak{g}} \to \mathbb{K}$ and the unit map $i\colon \mathbb{K} \to (W\mathfrak{g})^{\mathfrak{g}}$ are homotopy inverses. Since the Weil differential d vanishes on $(W\mathfrak{g})_{\mathrm{bas}} = (S\mathfrak{g}^*)^{\mathfrak{g}}$, any invariant polynomial $p \in (S\mathfrak{g}^*)^{\mathfrak{g}}$ of positive degree is a cocycle, and hence is the coboundary of an element in $(W\mathfrak{g})^{\mathfrak{g}}$.

DEFINITION 13.1. A *cochain of transgression* for $p \in (S\mathfrak{g}^*)^{\mathfrak{g}}$ is an odd element $C \in (W\mathfrak{g})^{\mathfrak{g}}$ with $\mathrm{d}(C) = p$.

The connection on $\wedge\mathfrak{g}^*$ gives a morphism of graded $\mathfrak{g}$-differential algebras $\pi\colon W\mathfrak{g} \to \wedge\mathfrak{g}^*$. It restricts to a morphism of differential algebras

$$\pi\colon (W\mathfrak{g})^{\mathfrak{g}} \to (\wedge\mathfrak{g}^*)^{\mathfrak{g}},$$

where $(\wedge\mathfrak{g}^*)^{\mathfrak{g}}$ carries the zero differential. Note that the space

$$(W^+\mathfrak{g})^{\mathfrak{g}} \cap \ker(\mathrm{d}) = (W\mathfrak{g})^{\mathfrak{g}} \cap \mathrm{ran}(\mathrm{d})$$

of cocycles of positive degree is mapped to $\mathrm{d}_\wedge(\wedge\mathfrak{g}^*)^{\mathfrak{g}} = 0$ under π.

PROPOSITION 13.2. *There is a well-defined linear map*

$$(S^+\mathfrak{g}^*)^{\mathfrak{g}} \to (\wedge\mathfrak{g}^*)^{\mathfrak{g}}, \quad p \mapsto \eta^p \tag{77}$$

such that $\eta^p = \pi(C)$ for any cochain of transgression C with $p = d(C)$. If p has degree $r > 0$, then η^p has degree $2r-1$. The map (77) vanishes on the subspace $((S^+\mathfrak{g}^*)^{\mathfrak{g}})^2$ spanned by products of invariant polynomials.

PROOF. Since $(W\mathfrak{g})^{\mathfrak{g}}$ is acyclic, and since the Weil differential vanishes on $(S\mathfrak{g}^*)^{\mathfrak{g}}$, any invariant polynomial $p \in (S^r\mathfrak{g}^*)^{\mathfrak{g}}$ of degree $r > 0$ is a coboundary. Hence we may write $p = \mathrm{d}(C)$ for some cochain of transgression $C \in (W^{2r-1}\mathfrak{g})^{\mathfrak{g}}$. If C' is another cochain of transgression for p, then $C' - C$ is closed, hence $\pi(C' - C) = 0$ as remarked above. Suppose $p, q \in (S\mathfrak{g}^*)^{\mathfrak{g}}$ have positive degree, and let $C \in \mathcal{T}$ be a cochain of transgression for p. Then qC^p is a cochain of transgression for qp. Since π is an algebra morphism, $\pi(qC) = \pi(q)\pi(C) = 0$. $\square$

CHAPTER 7

Quantum Weil algebras

We had seen that the Clifford algebra $\mathrm{Cl}(V)$ of a Euclidean vector space can be regarded as a quantization of the exterior algebra $\wedge(V)$, and similarly the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra as a quantization of the symmetric algebra $S(\mathfrak{g})$. In this section, we will study a similar quantization of the Weil algebra $W\mathfrak{g}$, for any Lie algebra $\mathfrak{g}$ with a non-degenerate invariant inner product B. Identifying $W\mathfrak{g} = S(\mathfrak{g}) \otimes \wedge(\mathfrak{g})$, where $S(\mathfrak{g})$ is the symmetric algebra generated by the 'curvature variables', this quantum Weil algebra is a $\mathfrak{g}$-differential algebra $\mathcal{W}\mathfrak{g} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$. The main result of this section (due to [**4, 5**]) is the existence of an isomorphism of $\mathfrak{g}$-differential spaces $W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$. On basic subcomplexes, this *quantization map* restricts to Duflo's isomorphism between the center of the enveloping algebra and invariants in $S\mathfrak{g}$. We further obtain a proof of Duflo's theorem for the case of quadratic Lie algebras, stating that this isomorphism respects product structures.

1. The $\mathfrak{g}$-differential algebra $\mathrm{Cl}(\mathfrak{g})$

Suppose the Lie algebra $\mathfrak{g}$ carries an invariant quadratic form B, used to identify $\mathfrak{g}^* \cong \mathfrak{g}$. The Lie bracket on $\mathfrak{g}$ will be denoted $[\cdot,\cdot]_\mathfrak{g}$, to avoid confusion with commutators in the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$. Recall that B defines a Poisson bracket on $\wedge(\mathfrak{g})$, given on generators by $\{\xi,\zeta\} = 2B(\xi,\zeta)$, and that $\iota(\xi) = \frac{1}{2}\{\xi,\cdot\}$. Define $\lambda(\xi) \in \wedge^2\mathfrak{g}$ and $\phi \in \wedge^3\mathfrak{g}$ by

$$\{\xi, \lambda(\zeta)\} = [\xi,\zeta]_\mathfrak{g}, \quad \{\xi,\phi\} = 2\lambda(\xi).$$

Thus $\lambda(\xi) = \lambda(\mathrm{ad}_\xi)$ in the notation of (24), while ϕ is the *structure constants tensor*.

Recall from Section §6.7 that $\wedge(\mathfrak{g}) \cong \wedge(\mathfrak{g}^*)$ carries the structure of a $\mathfrak{g}$-differential algebra. Observe that $\mathrm{d}\xi = 2\lambda(\xi)$, since

$$\iota(\zeta)\mathrm{d}\xi = L(\zeta)\xi = [\zeta,\xi]_\mathfrak{g} = \{\zeta,\lambda(\xi)\} = 2\iota(\zeta)\lambda(\xi).$$

Proposition 1.1. *The differential, Lie derivatives and contractions on $\wedge\mathfrak{g}$ are Poisson brackets:*

$$d = \{\phi,\cdot\}, \quad \iota(\xi) = \tfrac{1}{2}\{\xi,\cdot\}, \quad L(\xi) = \{\lambda(\xi),\cdot\}.$$

The elements $\xi, \lambda(\xi), \phi$ satisfy the following Poisson bracket relations:

$$\begin{aligned}
\{\phi, \phi\} &= 0, \\
\{\phi, \xi\} &= 2\lambda(\xi), \\
\{\phi, \lambda(\xi)\} &= 0, \\
\{\lambda(\xi), \lambda(\zeta)\} &= \lambda([\xi, \zeta]_{\mathfrak{g}}), \\
\{\lambda(\xi), \zeta\} &= [\xi, \zeta]_{\mathfrak{g}}, \\
\{\xi, \zeta\} &= 2\, B(\xi, \zeta).
\end{aligned}$$

Proof. For the first part it suffices to check on generators $\zeta \in \mathfrak{g} = \wedge^1 \mathfrak{g}$, using that both sides are derivations of $\wedge \mathfrak{g}$. But $L(\xi)\zeta = \{\lambda(\xi), \zeta\}$, $\iota(\xi)\zeta = \frac{1}{2}\{\xi, \zeta\}$ and $\mathrm{d}\zeta = 2\lambda(\zeta) = \{\phi, \zeta\}$ all follow from the definition.

We have $\{\xi, \zeta\} = 2B(\xi, \zeta)$ by definition of the Poisson bracket. The identity $L(\xi) = \{\lambda(\xi), \cdot\}$ determines all Poisson brackets with elements $\lambda(\xi)$, while $\{\phi, \xi\} = \lambda(\xi)$ is the definition of ϕ. The remaining bracket $\{\phi, \phi\} = \mathrm{d}\phi$ vanishes since $\phi \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$ is invariant, and hence is a cocycle for the Lie algebra differential. $\square$

Let us spell out the formulas in a basis e_a of $\mathfrak{g}$, with B-dual basis e^a. We have (cf. (26))

$$\lambda(\xi) = \frac{1}{4} \sum_a [\xi, e_a]_{\mathfrak{g}} \wedge e^a = -\frac{1}{4} \sum_{ab} B(\xi, [e_a, e_b]_{\mathfrak{g}})\, e^a \wedge e^b, \tag{78}$$

Likewise,

$$\phi = \frac{1}{3} \sum_a \lambda(e_a) \wedge e^a = -\frac{1}{12} \sum_{abc} B([e_a, e_b]_{\mathfrak{g}}, e_c)\, e^a \wedge e^b \wedge e^c. \tag{79}$$

The Poisson brackets from Proposition 1.1 quantize to commutators $[\cdot, \cdot]_{\mathrm{Cl}}$ in the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$. Let $\gamma(\xi) = q(\lambda(\xi))$. In the basis,

$$\gamma(\xi) = -\frac{1}{4} \sum_{ab} B(\xi, [e_a, e_b]_{\mathfrak{g}})\, e^a e^b$$

where the product on the right hand side is taken in the Clifford algebra.

Proposition 1.2. *The elements $q(\phi), \gamma(\xi), \xi$ in $\mathrm{Cl}(\mathfrak{g})$ satisfy the following commutation relations:*

$$\begin{aligned}
[q(\phi), q(\phi)]_{\mathrm{Cl}} &= \frac{1}{12} \operatorname{tr}(\operatorname{ad}(\mathrm{Cas}_{\mathfrak{g}})) \\
[q(\phi), \xi]_{\mathrm{Cl}} &= 2\gamma(\xi), \\
[q(\phi), \gamma(\xi)]_{\mathrm{Cl}} &= 0, \\
[\gamma(\xi), \zeta]_{\mathrm{Cl}} &= [\xi, \zeta]_{\mathfrak{g}} \\
[\gamma(\xi), \gamma(\zeta)]_{\mathrm{Cl}} &= \gamma([\xi, \zeta]_{\mathfrak{g}}), \\
[\xi, \zeta]_{\mathrm{Cl}} &= 2B(\xi, \zeta).
\end{aligned}$$

Here
$$\mathrm{Cas}_{\mathfrak{g}} = \sum_a e_a e^a \in U(\mathfrak{g})$$
is the quadratic Casimir element, and $\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}}))$ *is its trace in the adjoint representation on* $\mathfrak{g}$.

PROOF. The commutators with $\gamma(\xi)$ all follow from $L(\xi) = [\gamma(\xi), \cdot]_{\mathrm{Cl}}$, while the commutators with ξ are obtained from $\iota(\xi) = \frac{1}{2}[\xi, \cdot]_{\mathrm{Cl}}$. It remains to compute $[q(\phi), q(\phi)]_{\mathrm{Cl}} = 2q(\phi)^2$. Observe that it is a scalar, since
$$[\xi, [q(\phi), q(\phi)]_{\mathrm{Cl}}]_{\mathrm{Cl}} = 2[[\xi, q(\phi)]_{\mathrm{Cl}}, q(\phi)]_{\mathrm{Cl}} = 4[\gamma(\xi), q(\phi)]_{\mathrm{Cl}} = 0.$$
To find this scalar, recall that by Proposition 2.21 in §2, the square of $q(\phi)$ may be computed by applying the operator $\exp(-1/2\sum_a \iota(e^a) \otimes \iota(e_a))$ to $\phi \otimes \phi \in \wedge\mathfrak{g} \otimes \wedge\mathfrak{g}$, followed by the wedge product $\wedge\mathfrak{g} \otimes \wedge\mathfrak{g} \to \wedge\mathfrak{g}$, followed by q. Since we already know that only the scalar term survives, we obtain
$$\begin{aligned}
q(\phi)q(\phi) &= \frac{1}{6}\Big(-\sum_a \iota(e^a) \otimes \iota(e_a)\Big)^3 (\phi \otimes \phi)\\
&= -\frac{1}{6}\sum_{abc}(\iota(e_a)\iota(e_b)\iota(e_c)\phi)(\iota(e^a)\iota(e^b)\iota(e^c)\phi)\\
&= -\frac{1}{24}\sum_{abc} B(e_a, [e_b, e_c]_{\mathfrak{g}})B(e^a, [e^b, e^c]_{\mathfrak{g}})\\
&= -\frac{1}{24}\sum_{bc} B([e_b, e_c]_{\mathfrak{g}}, [e^b, e^c]_{\mathfrak{g}})\\
&= \frac{1}{24}\sum_{bc} B(e_c, [e_b, [e^b, e^c]_{\mathfrak{g}}]_{\mathfrak{g}})\\
&= \frac{1}{24}\sum_{c} B(e_c, \mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}})e^c)\\
&= \frac{1}{24}\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}}))
\end{aligned}$$
(See also the computation in §2, Example 3.5.) □

REMARK 1.3. The observation that $q(\phi)$ squares to the scalar appears to be due to Kostant and Sternberg [**48**].

As a consequence, we have

COROLLARY 1.4. *The Clifford algebra* $\mathrm{Cl}(\mathfrak{g})$ *is a filtered* $\mathfrak{g}$*-differential algebra with differential, Lie derivatives and contractions given as*
$$d_{\mathrm{Cl}} = [q(\phi), \cdot]_{\mathrm{Cl}}, \quad L_{\mathrm{Cl}}(\xi) = [\gamma(\xi), \cdot]_{\mathrm{Cl}}, \quad \iota_{\mathrm{Cl}}(\xi) = \tfrac{1}{2}[\xi, \cdot]_{\mathrm{Cl}}$$

REMARK 1.5. Suppose $\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}}) \neq 0$ (e.g. $\mathfrak{g}$ is simple). Then $H(\mathrm{Cl}(\mathfrak{g}), \mathrm{d}) = 0$. This follows since $q(\phi)$ is invertible in this case. See [**4**] for details.

The quantization map $q\colon \wedge\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ intertwines the Lie derivatives and contractions, but does *not* intertwine the differentials:

Proposition 1.6. *The Lie algebra differential $d_\wedge$ on $\wedge\mathfrak{g} \cong \wedge\mathfrak{g}^*$ and the Clifford differential differ by contraction with twice the cubic element $\phi \in (\wedge^3\mathfrak{g})^\mathfrak{g}$:*

$$q^{-1} \circ d_{\mathrm{Cl}} \circ q = d_\wedge + 2\iota(\phi).$$

Proof. For $x \in \mathrm{Cl}(\mathfrak{g})$, let x^L be the operator of left multiplication by x, and x^R the operator of ($\mathbb{Z}_2$-graded) right multiplication. Thus

$$x^L(y) = xy, \quad x^R(y) = (-1)^{|y||x|}yx$$

for homogeneous elements $x, y \in \mathrm{Cl}(\mathfrak{g})$, and $x^L - x^R = [x, \cdot]_{\mathrm{Cl}}$. If $\xi \in \mathfrak{g}$ we have

$$q^{-1} \circ \xi^L \circ q = \epsilon(\xi) + \iota(\xi), \quad q^{-1} \circ \xi^R \circ q = \epsilon(\xi) - \iota(\xi).$$

(The first formula follows since both sides define representations of $\mathrm{Cl}(\mathfrak{g})$ on $\wedge\mathfrak{g}$, and these two representations agree on $1 \in \wedge\mathfrak{g}$. The second formula is obtained similarly, or using that $q^{-1} \circ (\xi^L - \xi_R) \circ q = [\xi, \cdot]_{\mathrm{Cl}} = 2\iota(\xi)$.) Let e_a be a basis of $\mathfrak{g}$ with B-dual basis e^a, and write $\phi = \frac{1}{6}\sum_{abc}\phi^{abc}e_a \wedge e_b \wedge e_c$ where $\phi^{abc} = -\frac{1}{2}B([e^a, e^b]_\mathfrak{g}, e^c)$. Then

$$\mathrm{d}_\wedge = \sum_{abc}\phi^{abc}\epsilon(e_a)\epsilon(e_b)\iota(e_c),$$

Since $\iota(\xi)$, $\epsilon(\xi)$ are dual relative to the metric, the dual of $\mathrm{d}_\wedge$ is

$$\delta_\wedge = -\sum_{abc}\phi^{abc}\epsilon(e_a)\iota(e_b)\iota(e_c).$$

We compute,

$$\begin{aligned}
q^{-1} \circ q(\phi)^L \circ q &= \frac{1}{6}\sum_{abc}\phi^{abc}(\epsilon(e_a) + \iota(e_a))(\epsilon(e_b) + \iota(e_b))(\epsilon(e_c) + \iota(e_c)) \\
&= \epsilon(\phi) + \frac{1}{2}\sum_{abc}\phi^{abc}\big(\epsilon(e_a)\iota(e_b)\iota(e_c) + \epsilon(e_a)\epsilon(e_b)\iota(e_c)\big) + \iota(\phi) \\
&= \epsilon(\phi) + \iota(\phi) + \tfrac{1}{2}(\mathrm{d}_\wedge - \delta_\wedge), \\
q^{-1} \circ q(\phi)^R \circ q &= \epsilon(\phi) + \frac{1}{2}\sum_{abc}\phi^{abc}\big(\epsilon(e_a)\iota(e_b)\iota(e_c) - \epsilon(e_a)\epsilon(e_b)\iota(e_c)\big) \\
&= \epsilon(\phi) - \iota(\phi) - \tfrac{1}{2}(\mathrm{d}_\wedge + \delta_\wedge).
\end{aligned}$$

Subtracting the two results, we obtain

$$q^{-1} \circ \mathrm{d}_{\mathrm{Cl}} \circ q = \mathrm{d}_\wedge + 2\iota(\phi).$$

□

Proposition 1.7. *If $\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_\mathfrak{g})) \neq 0$, the cohomology of $\mathrm{Cl}(\mathfrak{g})$ is equal to zero.*

PROOF. Let $q(\phi)$ act on $\mathrm{Cl}(\mathfrak{g})$ by multiplication from the left. If $x \in \mathrm{Cl}(\mathfrak{g})$, then

$$\mathrm{d}(q(\phi)x) + q(\phi)\mathrm{d}x = (\mathrm{d}q(\phi))x = [q(\phi), q(\phi)]x = \frac{1}{12}\,\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}}))x.$$

Hence $h := \frac{12}{\mathrm{tr}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}}))} q(\phi)$ is a homotopy operator between the identity map and the zero map. ☐

2. The quantum Weil algebra

2.1. Poisson structure on the Weil algebra. Suppose $\mathfrak{g}$ carries an invariant non-degenerate symmetric bilinear form B, used to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and hence $W\mathfrak{g} = S(\mathfrak{g}) \otimes \wedge(\mathfrak{g})$. In §6, Remark 9.7, we introduced a family of $\mathfrak{g}$-differential algebras $(W\mathfrak{g})^{(r)}$ with contractions, Lie derivatives and differential

$$\iota(\xi)\zeta = B(\xi, \zeta), \quad \iota(\xi)\widehat{\zeta} = 0,$$
$$L(\xi)\zeta = [\xi, \zeta]_{\mathfrak{g}}, \quad L(\xi)\widehat{\zeta} = \widehat{[\xi, \zeta]_{\mathfrak{g}}},$$
$$\mathrm{d}^{(r)}\zeta = r\widehat{\zeta} + 2\lambda(\zeta), \quad \mathrm{d}^{(r)}\widehat{\zeta} = \sum_i \widehat{L(e_i)\zeta} e^i$$

(recall $\mathrm{d}_\wedge \xi = 2\lambda(\xi)$). We observed that for $r \neq 0$, these are all isomorphic by a simple rescaling of the $\hat{\zeta}$-variable. Both $S\mathfrak{g}, \wedge\mathfrak{g}$ are graded Poisson algebras, where the bracket on $S\mathfrak{g}$ is defined by the Lie bracket and that on $\wedge\mathfrak{g}$ is determined by B. Hence $W\mathfrak{g}$ becomes a Poisson algebra. Explicitly, the Poisson brackets of the generators are

$$\{\xi, \zeta\} = 2B(\xi, \zeta), \quad \{\widehat{\xi}, \widehat{\zeta}\} = \widehat{[\xi, \zeta]_{\mathfrak{g}}}, \quad \{\widehat{\xi}, \zeta\} = 0.$$

For the rest of this Section we will make the choice $r = 2$, since this is the unique choice for which the differential becomes a Poisson bracket:

LEMMA 2.1. *The derivation $d^{(r)}$ can be written as a Poisson bracket $\{D, \cdot\}$ if and only if $r = 2$. In fact, $d^{(2)} = \{D, \cdot\}$ where $D \in (W\mathfrak{g})^3$ is the cubic element*

$$D = \sum_i \widehat{e^i} e_i + \phi. \tag{80}$$

PROOF. Suppose $\mathrm{d}^{(r)} = \{D, \cdot\}$ for some element $D \in W\mathfrak{g}$. Since d raises degree by 1, while $\{\cdot, \cdot\}$ has degree -2, we can take D of degree 2. Since

$$\mathrm{d}^{(r)} e_j = r\hat{e}_j + 2\lambda(e_j),$$

and recalling $\{\phi, \xi\} = 2\lambda(\xi)$, we see that we must have

$$D = \frac{r}{2} \sum_i \hat{e}_i e^i + \phi.$$

But then $\{D, \hat{e}_j\}$ equals $\mathrm{d}^{(r)}\hat{e}_j = \widehat{[e_i, e_j]_{\mathfrak{g}}} e^i$ if and only if $r = 2$. ☐

For $r = 2$, the variables

$$\xi, \quad \overline{\xi} = \mathrm{d}^{(2)}\xi = 2(\widehat{\xi} + \lambda(\xi)).$$

satisfy the bracket relations

$$\{\xi, \zeta\} = 2B(\xi, \zeta), \quad \{\overline{\xi}, \overline{\zeta}\} = 2\overline{[\xi, \zeta]_{\mathfrak{g}}}, \quad \{\overline{\xi}, \zeta\} = 2[\xi, \zeta]_{\mathfrak{g}}.$$

In particular

$$\iota(\xi) = \tfrac{1}{2}\{\xi, \cdot\}, \quad L(\xi) = \tfrac{1}{2}\{\overline{\xi}, \cdot\}.$$

REMARK 2.2. (1) For general $r \neq 0$, the bracket relation among the variables $\overline{\xi} = \mathrm{d}^{(r)}\xi = r\widehat{\xi} + 2\lambda(\xi)$ takes on the more complicated form

$$\{\overline{\xi}, \overline{\zeta}\} = r\overline{[\xi, \zeta]_{\mathfrak{g}}} + 2(2 - r)\lambda([\xi, \zeta]_{\mathfrak{g}}).$$

(2) Rather than working with the modified differential $\mathrm{d}^{(2)}$, one can also work with the Poisson bracket on $\wedge\mathfrak{g}$ and Clifford algebra structure defined by $\frac{1}{2}B$. This is the approach taken in [**4**].

For the rest of this chapter, we put $r = 2$. The element $D \in (W\mathfrak{g})^3$ (cf. (80)) reads, in terms the variables $\xi, \overline{\xi}$,

$$D = \tfrac{1}{2}\sum_i \overline{e}_i e^i - 2\phi.$$

PROPOSITION 2.3. *We have the following Poisson bracket relations in* $W\mathfrak{g}$*:*

$$\begin{aligned} \{D, D\} &= 2\textstyle\sum_i \widehat{e_i}\widehat{e^i}, \\ \{D, \xi\} &= \overline{\xi}, \\ \{D, \overline{\xi}\} &= 0 \\ \{\overline{\xi}, \zeta\} &= 2[\xi, \zeta]_{\mathfrak{g}} \\ \{\overline{\xi}, \overline{\zeta}\} &= 2\overline{[\xi, \zeta]_{\mathfrak{g}}} \\ \{\xi, \zeta\} &= 2B(\xi, \zeta). \end{aligned}$$

In particular, $\{D, \cdot\} = d^{(2)}$.

PROOF. The bracket relations involving $\xi, \overline{\xi}$ are all simple consequences of $\{\xi, \cdot\} = 2\iota(\xi)$ and $\{\overline{\xi}, \cdot\} = 2L(\xi)$. It remains to check the formula for $\{D, D\}$. We have

$$\{D, D\} = 2\sum_{ij} \widehat{e^i}\widehat{e^j} B(e_i, e_j) + \ldots = 2\sum_i \widehat{e^i}\widehat{e}_i + \ldots$$

where the dots indicate terms in $S\mathfrak{g}\otimes\wedge^+ g$. But $\{D, D\}$ lies in the symmetric algebra generated by the $\widehat{\xi}$, since

$$\iota(\xi)\{D, D\} = \tfrac{1}{2}\{\xi, \{D, D\}\} = \{\{\xi, D\}, D\} = \{\overline{\xi}, D\} = 0.$$

Hence the ... terms all cancel. $\square$

REMARK 2.4. Note that the quadratic Casimir element $\sum_i \widehat{e_i}\widehat{e^i}$ lies in the Poisson center of $W\mathfrak{g}$. Hence, the formula for $\{D, D\}$ is consistent with $[\mathrm{d}^{(2)}, \mathrm{d}^{(2)}] = 0$.

2.2. Definition of the quantum Weil algebra. Quantizing the Poisson bracket relations of $W\mathfrak{g}$, we arrive at the following definition.

DEFINITION 2.5. The *quantum Weil algebra* $\mathcal{W}\mathfrak{g}$ is the filtered super algebra, generated by odd elements ξ of filtration degree 1 and even elements $\overline{\xi}$ of filtration degree 2, with commutator relations

$$[\xi, \zeta]_{\mathcal{W}} = 2B(\xi, \zeta), \quad [\overline{\xi}, \zeta]_{\mathcal{W}} = 2[\xi, \zeta]_{\mathfrak{g}}, \quad [\overline{\xi}, \overline{\zeta}]_{\mathcal{W}} = 2\overline{[\xi, \zeta]_{\mathfrak{g}}}.$$

That is, $\mathcal{W}\mathfrak{g}$ is the *semi-direct product*

$$\mathcal{W}\mathfrak{g} = U\mathfrak{g} \otimes_s \mathrm{Cl}(\mathfrak{g}),$$

where $U\mathfrak{g}$ is the enveloping algebra generated by the elements $\frac{1}{2}\overline{\xi}$, acting on $\mathrm{Cl}(\mathfrak{g})$ by the extension of the adjoint representation.

Put differently, $\mathcal{W}\mathfrak{g}$ is the quotient of $\tilde{W}\mathfrak{g} = T(E_{\mathfrak{g}}[1])$ by the two-sided ideal generated by elements of the form,

$$\begin{aligned}
&\xi \otimes \zeta + \zeta \otimes \xi - 2B(\xi, \zeta)\\
&\overline{\xi} \otimes \overline{\zeta} + \overline{\zeta} \otimes \overline{\xi} - 2\overline{[\xi, \zeta]_{\mathfrak{g}}}\\
&\overline{\xi} \otimes \zeta - \zeta \otimes \overline{\xi} - [\xi, \zeta]_{\mathfrak{g}}.
\end{aligned}$$

It is straightforward to see that this ideal is invariant under the differential, contractions and Lie derivatives on $\tilde{W}\mathfrak{g}$. Hence, $\mathcal{W}\mathfrak{g}$ carries a unique structure of a filtered $\mathfrak{g}$-differential algebra, such that the quotient map $\tilde{W}\mathfrak{g} \to \mathcal{W}\mathfrak{g}$ is a morphism of $\mathfrak{g}$-differential algebras. The formulas for differential, contractions and Lie derivatives are induced from those on $\tilde{W}\mathfrak{g}$, and are given on generators by

$$\begin{aligned}
\mathrm{d}\xi &= \overline{\xi}, & \mathrm{d}\overline{\xi} &= 0,\\
\iota(\xi)\zeta &= B(\xi, \zeta), & \iota(\xi)\overline{\zeta} &= [\xi, \zeta]_{\mathfrak{g}},\\
L(\xi)\zeta &= [\xi, \zeta]_{\mathfrak{g}}, & L(\xi)\overline{\zeta} &= \overline{[\xi, \zeta]_{\mathfrak{g}}}.
\end{aligned}$$

As for the commutative Weil algebra $W\mathfrak{g}$, it is often convenient to change the variables to ξ, $\widehat{\xi} = \frac{1}{2}\overline{\xi} - \gamma(\xi)$ (where $\gamma(\xi) = q(\lambda(\xi))$). In terms of the new generators the commutator relations read,

$$[\xi, \zeta]_{\mathcal{W}} = 2B(\xi, \zeta), \quad [\widehat{\xi}, \zeta]_{\mathcal{W}} = 0 \quad [\widehat{\xi}, \widehat{\zeta}]_{\mathcal{W}} = \widehat{[\xi, \zeta]_{\mathfrak{g}}}.$$

Hence

$$\mathcal{W}\mathfrak{g} = U\mathfrak{g} \otimes \mathrm{Cl}\mathfrak{g},$$

where $U\mathfrak{g}$ denotes the enveloping algebra generated by the elements $\widehat{\xi}$. As for the usual Weil algebra $W\mathfrak{g}$, we see that the basic subcomplex is

$$(\mathcal{W}\mathfrak{g})_{\mathrm{bas}} = (U\mathfrak{g})^{\mathfrak{g}},$$

using the enveloping algebra generated by the $\widehat{\xi}$'s. Since it has no odd component, the differential on the basic subcomplex is zero. Thus

$$H_{\mathrm{bas}}(\mathcal{W}\mathfrak{g}) = (U\mathfrak{g})^{\mathfrak{g}}.$$

In terms of the new variables, it is also clear that the associated graded algebra to the filtered algebra $\mathcal{W}\mathfrak{g}$ is

$$\mathrm{gr}(\mathcal{W}\mathfrak{g}) = W\mathfrak{g},$$

because $\mathrm{gr}(U\mathfrak{g}) = S\mathfrak{g}$ and $\mathrm{gr}(\mathrm{Cl}\mathfrak{g}) = \wedge\mathfrak{g}$. From the formulas on generators, we see that the Poisson bracket on $W\mathfrak{g}$ is induced from the commutator on $\mathcal{W}\mathfrak{g}$. In this sense, $\mathcal{W}\mathfrak{g}$ is a quantization of $W\mathfrak{g}$.

2.3. The cubic Dirac operator. Let $\mathcal{D} \in \mathcal{W}^{(3)}\mathfrak{g}$ be the 'quantized version' of the element $D \in W^3\mathfrak{g}$,

$$\mathcal{D} = \tfrac{1}{2}\sum_i \overline{e^i} e_i - 2q(\phi).$$

In terms of the variables $\xi, \widehat{\xi}$, the cubic Dirac operator takes on the form,

$$\mathcal{D} = \sum_i \widehat{e}^i e_i + q(\phi).$$

Similar to Proposition 2.3, we have:

THEOREM 2.6. *We have the following commutator relations in $\mathcal{W}\mathfrak{g}$:*

$$\begin{aligned}
[\mathcal{D}, \mathcal{D}]_{\mathcal{W}} &= 2\mathrm{Cas}_{\mathfrak{g}} + \frac{1}{12}\,\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}}),\\
[\overline{\xi}, \mathcal{D}]_{\mathcal{W}} &= 0,\\
[\xi, \mathcal{D}]_{\mathcal{W}} &= \overline{\xi}\\
[\overline{\xi}, \zeta]_{\mathcal{W}} &= 2[\xi, \zeta]_{\mathfrak{g}}\\
[\overline{\xi}, \overline{\zeta}]_{\mathcal{W}} &= 2\overline{[\xi, \zeta]_{\mathfrak{g}}}\\
[\xi, \zeta]_{\mathcal{W}} &= 2B(\xi, \zeta)
\end{aligned}$$

Here $\mathrm{Cas}_{\mathfrak{g}} = \sum_i \widehat{e_i}\widehat{e^i} \in U\mathfrak{g}$ *is the Casimir element, and* $\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}})$ *its trace in the adjoint representation. The contractions, Lie derivatives and differential on $\mathcal{W}\mathfrak{g}$ are all inner derivations:*

$$\iota(\xi) = \tfrac{1}{2}[\xi, \cdot]_{\mathcal{W}}, \quad L(\xi) = \tfrac{1}{2}[\overline{\xi}, \cdot]_{\mathcal{W}}, \quad d = [\mathcal{D}, \cdot]_{\mathcal{W}}. \tag{81}$$

PROOF. The identities $\iota(\xi) = \frac{1}{2}[\xi, \cdot]_{\mathcal{W}}$, $L(\xi) = \frac{1}{2}[\overline{\xi}, \cdot]_{\mathcal{W}}$ are clear from the definition of $\mathcal{W}\mathfrak{g}$. They imply all the commutator relations involving $\xi, \overline{\xi}$. From $[\xi, [\mathcal{D}, \mathcal{D}]] = 2[\overline{\xi}, \mathcal{D}] = 0$, we see that $[\mathcal{D}, \mathcal{D}] \in U\mathfrak{g}$ (the enveloping algebra generated by the variables $\widehat{\xi}$). Denoting terms in $U\mathfrak{g} \otimes q(\wedge^+\mathfrak{g})$ by

..., and using our earlier computation of $[q(\phi), q(\phi)]$, we find:

$$\begin{aligned}[\mathcal{D},\mathcal{D}] &= \sum_{ij}[\widehat{e_i}e^i + q(\phi), \widehat{e_j}e^j + q(\phi)] \\ &= \sum_{ij}\widehat{e_i}\widehat{e_j}[e^i, e^j]_{\mathrm{Cl}} + [q(\phi), q(\phi)] + \dots \\ &= 2\,\mathrm{Cas}_{\mathfrak{g}} + \frac{1}{12}\,\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}}),\end{aligned}$$

since the terms in $U\mathfrak{g}\otimes q(\wedge^+\mathfrak{g})$ must cancel. The identity $\mathrm{d} = [\mathcal{D},\cdot]_{\mathcal{W}}$ follows since both sides are derivations that agree on generators. $\square$

The element $\mathcal{D}$ is called the *cubic Dirac operator*, after Kostant [**45**]. It is an algebraic Dirac operator in the sense that its square

$$\mathcal{D}^2 = \mathrm{Cas}_{\mathfrak{g}} + \frac{1}{24}\,\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}})$$

is the quadratic Casimir (viewed as the algebraic counterpart to a Laplacian), up to lower order terms. In §9 below, we will discuss interpretations of $\mathcal{D}$ as geometric Dirac operators over Lie groups.

2.4. $\mathcal{W}\mathfrak{g}$ as a level 1 enveloping algebra. We had remarked that the Clifford algebra can be viewed as a 'level 1 enveloping algebra' of a super Lie algebra, see §5.1.9. The quantum Weil algebra $\mathcal{W}\mathfrak{g}$ can be viewed similarly, using the graded differential Lie algebra $E_{\mathfrak{g}}[1] = \mathfrak{g}[1]\rtimes\mathfrak{g}$. For $\xi\in\mathfrak{g}$ let I_ξ, L_ξ denote the corresponding generators of degree $-1, 0$. The bilinear form B defines a central extension

$$\mathbb{K}[2]\oplus\mathfrak{g}[1]\rtimes\mathfrak{g}, \tag{82}$$

where $\mathbb{K}[2]$ is spanned by a central generator c of degree -2, and with the new bracket relations

$$[I_\xi, I_\zeta] = 2B(\xi,\zeta)\mathsf{c},\quad [L_\xi, I_\zeta] = I_{[\xi,\zeta]_{\mathfrak{g}}},\quad [L_\xi, L_\zeta] = L_{[\xi,\zeta]_{\mathfrak{g}}}.$$

It is a graded $\mathfrak{g}$-differential Lie algebra, where the action of contractions and Lie derivatives is just the adjoint action of $\mathfrak{g}[1]\rtimes\mathfrak{g}$, and where $\mathrm{d}I_\xi = L_\xi,\ \mathrm{d}L_\xi = 0,\ \mathrm{d}\mathsf{c} = 0$. After degree shift by 2, (82) is a *filtered* differential Lie algebra,

$$\mathbb{K}\oplus\mathfrak{g}[-1]\rtimes\mathfrak{g}[-2],$$

where c now has filtration degree 0. We may regard $\mathcal{W}\mathfrak{g}$ as the level 1 enveloping algebra,

$$\mathcal{W}\mathfrak{g} = U(\mathbb{K}\oplus\mathfrak{g}[-1]\rtimes\mathfrak{g}[-2])/ < \mathsf{c}-1 >,$$

(with the internal filtration). The generators $\xi, \overline{\xi}$ correspond to $I_\xi, 2L_\xi$ with the shifted filtration degrees.

One can also go one step further and consider the graded differential Lie algebra

$$\mathbb{K}[2]\oplus\mathfrak{g}[1]\rtimes\mathfrak{g}\oplus\mathbb{K}[-1], \tag{83}$$

containing (82) as a subalgebra, and with the generator $D \in \mathbb{K}[-1]$ acting as the differential:

$$[D, D] = 0, \quad [D, \mathsf{c}] = 0, \quad [D, I_\xi] = L_\xi, \quad [D, L_\xi] = 0.$$

It is a graded super Lie algebra, with an odd invariant bilinear form given by $\langle I_\xi,\, L_\zeta\rangle = B(\xi, \zeta)$, $\langle \mathsf{c},\, D\rangle = 1$ (all other inner products among generators are 0).

Remark 2.7. The double extension (83) is similar to the standard double extension of the loop algebra of a quadratic Lie algebra. Indeed, both are obtained as double extensions for an orthogonal derivation [**5**]. See Ševera [**57**] for a more conceptual explanation of this relationship.

The Lie algebra (83) and the corresponding super Lie group appear in the recent work of Li-Bland and Ševera [**23**].

3. Application: Duflo's theorem

Using the quantum Weil algebra, we will now prove Duflo's theorem (cf. §5, Theorem 6.4) for the case of quadratic Lie algebras $\mathfrak{g}$, following [**4, 5**]. Use the bilinear form on $\mathfrak{g}$ to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Think of $W\mathfrak{g}$ (with differential $\mathrm{d}^{(2)}$) as $S(E_\mathfrak{g}[1])$, and let

$$q\colon W\mathfrak{g} \to \mathcal{W}\mathfrak{g} \tag{84}$$

be the map, extending $\xi \mapsto \xi$, $\overline{\xi} \mapsto \overline{\xi}$ on generators by super symmetrization. It is a vector space isomorphism, since its associated graded map is the identity map.

Remark 3.1. The quantization map q can also be viewed as the map defined by super symmetrization

$$S(\mathbb{K} \oplus \mathfrak{g}[-1] \rtimes \mathfrak{g}[-2]) \to U(\mathbb{K} \oplus \mathfrak{g}[-1] \rtimes \mathfrak{g}[-2])$$

after taking a quotient of both sides by the respective ideals $< \mathsf{c} - 1 >$.

By Proposition 5.2, $q\colon W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$ is an isomorphism of $\mathfrak{g}$-differential spaces, and hence it restricts to vector space isomorphisms

$$(W\mathfrak{g})_{\mathrm{hor}} = S\mathfrak{g} \to (\mathcal{W}\mathfrak{g})_{\mathrm{hor}} = U\mathfrak{g} \tag{85}$$

and hence

$$(S\mathfrak{g})^\mathfrak{g} \to (U\mathfrak{g})^\mathfrak{g}. \tag{86}$$

We may also think of (86) as the map in basic cohomology $H_{\mathrm{bas}}(W\mathfrak{g}) = (S\mathfrak{g})^\mathfrak{g} \to H_{\mathrm{bas}}(\mathcal{W}\mathfrak{g}) = (U\mathfrak{g})^\mathfrak{g}$, since the differential on the basic subcomplexes is zero.

theorem 3.2. *The map* (86) *is an isomorphism of algebras.*

PROOF. The symmetrization map factors through the non-commutative Weil algebra $\tilde{W}\mathfrak{g}$:

$$W\mathfrak{g} \to \tilde{W}\mathfrak{g} \to \mathcal{W}\mathfrak{g},$$

hence (86) factors as

$$(S\mathfrak{g})^{\mathfrak{g}} \to H_{\mathrm{bas}}(\tilde{W}\mathfrak{g}) \to (U\mathfrak{g})^{\mathfrak{g}}.$$

The second map is an algebra homomorphism since it is the map in basic cohomology induced by the homomorphism of $\mathfrak{g}$-differential algebras $\tilde{W}\mathfrak{g} \to \mathcal{W}\mathfrak{g}$. The first map is an algebra homomorphism, since it is inverse to the map in basic cohomology induced by the homomorphism of $\mathfrak{g}$-differential algebras $\tilde{W}\mathfrak{g} \to W\mathfrak{g}$. $\square$

We stress that the isomorphism $S\mathfrak{g} \to U\mathfrak{g}$ in (85) is *not* the symmetrization map for the enveloping algebra, even though it comes from the (super-)symmetrization map q of the Weil algebras $\mathcal{W}\mathfrak{g}$. Indeed, q is defined using symmetrization with respect to $\xi, \overline{\xi}$, but $U\mathfrak{g}$ is the enveloping algebra generated by the variables $\widehat{\xi}$. To get an explicit formula for (85), we want to express q in terms of the symmetrization map

$$\mathrm{sym} = \mathrm{sym}_U \otimes q_{\mathrm{Cl}} \colon S\mathfrak{g} \otimes \wedge\mathfrak{g} \to U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g})$$

relative to $\xi, \widehat{\xi}$.

Recall that the formula relating exponentials in the exterior and in the Clifford algebra (cf. §4, Theorem 3.8) involved a smooth function

$$\mathcal{S} \colon \mathfrak{g} \to \wedge\mathfrak{g}$$

of the form $\mathcal{S}(\xi) = J^{1/2}(\xi)\exp(\mathfrak{r}(\xi))$ where $J^{1/2}$ is the 'Duflo factor' and $\mathfrak{r}$ is a certain meromorphic function with values in $\wedge^2\mathfrak{g}$. This function gives rise to an element

$$\widetilde{\mathcal{S}} \in \overline{S}\mathfrak{g}^* \otimes \wedge\mathfrak{g},$$

where the first factor can be thought of as constant coefficient (infinite order) differential operators. This element acts on $W\mathfrak{g} = S\mathfrak{g} \otimes \wedge\mathfrak{g}$ in a natural way: The $\overline{S}\mathfrak{g}^*$ factor acts as an infinite order differential operator, while the second factor acts by contraction.

THEOREM 3.3. *The isomorphism of $\mathfrak{g}$-differential spaces $q \colon W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$ is given in terms of the generators $\xi, \widehat{\xi}$ by*

$$q = \mathrm{sym} \circ \widetilde{\mathcal{S}} \colon S\mathfrak{g} \otimes \wedge\mathfrak{g} \to U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g}).$$

In particular, its restriction to $S\mathfrak{g} \subset W\mathfrak{g}$ is the Duflo map,

$$\mathrm{Duf} = \mathrm{sym} \circ \widetilde{J^{1/2}} \colon S\mathfrak{g} \to U\mathfrak{g}.$$

PROOF. By definition, q is the symmetrization map relative to the variables $\xi, \overline{\xi}$. It may be characterized as follows: For all odd variables $\nu^i \in \mathfrak{g}^*$ and all even variables $\mu^j \in \mathfrak{g}^*$, and all $N = 0, 1, 2 \ldots$,

$$q\Big(\big(\sum_i \nu^i e_i + \tfrac{1}{2}\sum_j \mu^j \overline{e}_j\big)^N\Big) = \big(\sum_i \nu^i e_i + \tfrac{1}{2}\sum_j \mu^j \overline{e}_j\big)^N.$$

These conditions may be summarized in a single condition,

$$q\Big(\exp_W(\sum_i \nu^i e_i + \tfrac{1}{2}\sum_j \mu^j \overline{e}_j)\Big) = \exp_{\mathcal{W}}(\sum_i \nu^i e_i + \tfrac{1}{2}\sum_j \mu^j \overline{e}_j),$$

to be interpreted as an equality of formal power series in the variables ν, μ.

We want to express q in terms of the generators $e_i, \widehat{e}_i = \frac{1}{2}\overline{e}_i - \lambda(e_i)$ of $W\mathfrak{g}$ respectively $e_i, \widehat{e}_i = \frac{1}{2}\overline{e}_i - \gamma(e_i)$ of $\mathcal{W}\mathfrak{g}$. Using that $\widehat{e}_i$ and e_j commute in $\mathcal{W}\mathfrak{g}$,

$$\exp_{\mathcal{W}}(\sum_i \nu^i e_i + \tfrac{1}{2}\sum_j \mu^j \overline{e}_j) = \exp_U(\sum_j \mu^j \widehat{e}_j)\exp_{\mathrm{Cl}}(\sum_i \nu^i e_i + \sum_j \mu^j \gamma(e_j)).$$

The first factor is $\mathrm{sym}\big(\exp_S(\sum_j \mu^j \widehat{e}_j)\big)$ by definition of the symmetrization map $\mathrm{sym}\colon S\mathfrak{g} \to U\mathfrak{g}$. The second factor is the Clifford exponential of a quadratic element, and is related to the corresponding exponential in the exterior algebra,

$$\exp_{\mathrm{Cl}}(\sum_i \nu^i e_i + \sum_j \mu^j \gamma(e_j)) = q_{\mathrm{Cl}}\Big(\iota(\mathcal{S}(\mu))\exp_\wedge(\sum_i \nu^i e_i + \sum_j \mu^j \lambda(e_j))\Big).$$

Hence

$$\begin{aligned}
\exp_{\mathcal{W}}&\Big(\sum_i \nu^i e_i + \tfrac{1}{2}\sum_j \mu^j \overline{e}_j)\\
&= \mathrm{sym}_W\Big(\iota(\mathcal{S}(\mu))\exp_S(\sum_j \mu^j \widehat{e}_j)\exp_\wedge(\sum_i \nu^i e_i + \sum_j \mu^j \lambda(e_j))\Big)\\
&= \mathrm{sym}_W \circ \widetilde{S}\Big(\exp_W\big(\sum_i \nu^i e_i + \sum_j \mu^j \lambda(e_j) + \sum_j \mu^j \widehat{e}_j\big)\Big)\\
&= \mathrm{sym}_W \circ \widetilde{S}\Big(\exp_W\big(\sum_i \nu^i e_i + \tfrac{1}{2}\sum_j \mu^j \overline{e}_j\big)\Big).
\end{aligned}$$

□

This completes our proof of Duflo's theorem in the case of quadratic Lie algebras. Note that in this proof, the 'Duflo factor' $J^{1/2}$ is naturally interpreted in terms of Clifford algebra computations. On the other hand, Duflo's theorem is valid for *arbitrary* Lie algebras (not only quadratic ones). In recent years, new proofs for the general case have been found using Kontsevich's theory of deformation quantization [**42**], and more recently by Alekseev-Torossian [**6**] in their approach to the Kashiwara-Vergne conjecture [**40**]. However, all of these proofs are a great deal more involved than the argument for the quadratic case.

4. Relative Dirac operators

We will now generalize to Kostant's [**45**] relative version of the cubic Dirac operator, associated to a pair of quadratic Lie algebras. Some of the ideas are also present in lecture notes by A. Wassermann [**61**]. The discussion uses simplifications from [**5**].

Suppose $\mathfrak{k} \subset \mathfrak{g}$ is a quadratic subalgebra, that is, the restriction of B to $\mathfrak{k}$ is non-degenerate. Let $\mathfrak{p} = \mathfrak{k}^\perp$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is a $\mathfrak{k}$-invariant orthogonal decomposition.

DEFINITION 4.1. The *relative Weil algebra* for the pair $\mathfrak{g}, \mathfrak{k}$ is the subalgebra

$$\mathcal{W}(\mathfrak{g}, \mathfrak{k}) = (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^{\mathfrak{k}-\mathrm{inv}}$$

of $\mathcal{W}\mathfrak{g}$.

Equivalently, viewing $\mathcal{W}\mathfrak{g}$ as a $\mathfrak{k}$-differential subalgebra by restriction, $\mathcal{W}(\mathfrak{g}, \mathfrak{k})$ is the $\mathfrak{k}$-basic subalgebra. Define an injective algebra homomorphism

$$j\colon \mathcal{W}\mathfrak{k} \to \mathcal{W}\mathfrak{g}, \tag{87}$$

by sending the generators ξ, $\overline{\xi} \in \mathcal{W}\mathfrak{k}$ for $\xi \in \mathfrak{k}$ to the corresponding generators in $\mathcal{W}\mathfrak{g}$:

$$j(\xi) = \xi,\ j(\overline{\xi}) = \overline{\xi}.$$

Checking on generators, it is immediate that j is a morphism of $\mathfrak{k}$-differential algebras.

PROPOSITION 4.2. *The subalgebra $\mathcal{W}(\mathfrak{g}, \mathfrak{k})$ is the commutant of $j(\mathcal{W}\mathfrak{k}) \subset \mathcal{W}\mathfrak{g}$.*

PROOF. Since $j(\mathcal{W}\mathfrak{k})$ has generators $j(\xi) = \xi$ and $j(\overline{\xi}) = \overline{\xi}$, for $\xi \in \mathfrak{k}$, its commutant consists of elements annhilated by all $\iota(\xi) = \frac{1}{2}[\xi, \cdot]$ and $L(\xi) = \frac{1}{2}[\overline{\xi}, \cdot]$ for $\xi \in \mathfrak{k}$. In other words, the commutant is the $\mathfrak{k}$-basic subalgebra $(\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{bas}} = \mathcal{W}(\mathfrak{g}, \mathfrak{k})$. ☐

REMARK 4.3. The morphism j can also be viewed as follows: The inclusion $\mathfrak{k} \hookrightarrow \mathfrak{g}$ gives a morphism of graded $\mathfrak{k}$-differential Lie algebras,

$$\mathbb{K}[2] \oplus (\mathfrak{k}[1] \rtimes \mathfrak{k}) \to \mathbb{K}[2] \oplus (\mathfrak{g}[1] \rtimes \mathfrak{g}).$$

Degree shift by 2 turns it into a morphism of filtered $\mathfrak{k}$-differential Lie algebras, and the enveloping functor gives a morphism of filtered $\mathfrak{k}$-differential algebras,

$$U(\mathbb{K} \oplus (\mathfrak{k}[-1] \rtimes \mathfrak{k}[-2])) \to U(\mathbb{K} \oplus (\mathfrak{g}[-1] \rtimes \mathfrak{g}[-2])).$$

The map of quantum Weil algebras is obtained by taking the quotient by the ideal $< \mathsf{c} - 1 >$ on both sides.

DEFINITION 4.4. The difference

$$\mathcal{D}(\mathfrak{g}, \mathfrak{k}) = \mathcal{D}_\mathfrak{g} - j(\mathcal{D}_\mathfrak{k})$$

is called the *relative Dirac operator* for the pair $\mathfrak{g}, \mathfrak{k}$.

THEOREM 4.5. *The relative Dirac operator $\mathcal{D}(\mathfrak{g},\mathfrak{k})$ lies in the differential subalgebra $\mathcal{W}(\mathfrak{g},\mathfrak{k})$, and the restriction of $[\mathcal{D}(\mathfrak{g},\mathfrak{k}),\cdot]$ to $\mathcal{W}(\mathfrak{g},\mathfrak{k})$ is the differential of $\mathcal{W}(\mathfrak{g},\mathfrak{k})$. The commutator of $\mathcal{D}(\mathfrak{g},\mathfrak{k})$ with itself is given by the formula,*

$$[\mathcal{D}(\mathfrak{g},\mathfrak{k}),\mathcal{D}(\mathfrak{g},\mathfrak{k})] = 2\,\mathrm{Cas}_{\mathfrak{g}} - 2j(\mathrm{Cas}_{\mathfrak{k}}) + \frac{1}{12}\,\mathrm{tr}_{\mathfrak{g}}\,\mathrm{Cas}_{\mathfrak{g}} - \frac{1}{12}\,\mathrm{tr}_{\mathfrak{k}}\,\mathrm{Cas}_{\mathfrak{k}}, \tag{88}$$

where $\mathrm{Cas}_{\mathfrak{g}} \in U(\mathfrak{g})$ and $\mathrm{Cas}_{\mathfrak{k}} \in U(\mathfrak{k})$ are the quadratic Casimir elements, for the enveloping algebra generated by the 'curvature variables'.

PROOF. The element $\mathcal{D}(\mathfrak{g},\mathfrak{k})$ is $\mathfrak{k}$-horizontal since

$$[\mathcal{D}(\mathfrak{g},\mathfrak{k}),\xi] = [\mathcal{D}_{\mathfrak{g}},\xi] - j([\mathcal{D}_{\mathfrak{k}},\xi]) = \overline{\xi} - j(\overline{\xi}) = 0$$

for all $\xi \in \mathfrak{k}$. Similarly, it is $\mathfrak{k}$-invariant. This shows $\mathcal{D}(\mathfrak{g},\mathfrak{k}) \in \mathcal{W}(\mathfrak{g},\mathfrak{k})$. Next, for $x \in \mathcal{W}(\mathfrak{g},\mathfrak{k})$ we have $[\mathcal{D}(\mathfrak{g},\mathfrak{k}),x] = [\mathcal{D}_{\mathfrak{g}},x] = \mathrm{d}x$, since elements of $\mathcal{W}(\mathfrak{g},\mathfrak{k})$ commute with elements of $\mathcal{W}\mathfrak{k}$.

Write $\mathcal{D}_{\mathfrak{g}} = j(\mathcal{D}_{\mathfrak{k}}) + \mathcal{D}(\mathfrak{g},\mathfrak{k})$. Since $j(\mathcal{D}_{\mathfrak{k}}) \in j(\mathcal{W}\mathfrak{k})$ and $\mathcal{D}(\mathfrak{g},\mathfrak{k}) \in \mathcal{W}(\mathfrak{g},\mathfrak{k})$ commute, we obtain

$$[\mathcal{D}_{\mathfrak{g}},\mathcal{D}_{\mathfrak{g}}] = j([\mathcal{D}_{\mathfrak{k}},\mathcal{D}_{\mathfrak{k}}]) + [\mathcal{D}(\mathfrak{g},\mathfrak{k}),\mathcal{D}(\mathfrak{g},\mathfrak{k})].$$

The formula for $[\mathcal{D}(\mathfrak{g},\mathfrak{k}),\mathcal{D}(\mathfrak{g},\mathfrak{k})]$ now follows from the known formulas for $[\mathcal{D}_{\mathfrak{g}},\mathcal{D}_{\mathfrak{g}}]$ and $[\mathcal{D}_{\mathfrak{k}},\mathcal{D}_{\mathfrak{k}}]$, see Theorem §5.2.6. □

Let us now given a formula for $\mathcal{D}(\mathfrak{g},\mathfrak{k})$ in terms of the variables ξ, $\widehat{\xi}$. Invariance of the bilinear form implies that $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$. Hence ad_{ξ}, $\xi \in \mathfrak{k}$ decomposes as a direct sum of its $\mathfrak{k}$ and $\mathfrak{p}$ components. Accordingly the element $\lambda_{\mathfrak{g}}(\xi) \in \wedge^2\mathfrak{g}$ decomposes as

$$\lambda_{\mathfrak{g}}(\xi) = \lambda_{\mathfrak{k}}(\xi) + \lambda_{\mathfrak{p}}(\xi),$$

for some $\lambda_{\mathfrak{p}}(\xi) = \lambda(\mathrm{ad}(\xi)|_{\mathfrak{p}}) \in \wedge^2\mathfrak{p}$, and hence by quantization

$$\gamma_{\mathfrak{g}}(\xi) = \gamma_{\mathfrak{k}}(\xi) + \gamma_{\mathfrak{p}}(\xi).$$

Commutator with $\gamma_{\mathfrak{p}}(\xi) = q(\lambda_{\mathfrak{p}}(\xi))$, $\xi \in \mathfrak{k}$ generates the adjoint action of $\mathfrak{k}$ on $\mathrm{Cl}(\mathfrak{p})$.

LEMMA 4.6. *The homomorphism $j\colon \mathcal{W}\mathfrak{k} \to \mathcal{W}\mathfrak{g}$ is given in terms of generators $\xi, \widehat{\xi}$ by*

$$j(\xi) = \xi, \quad j(\widehat{\xi}) = \widehat{\xi} + \gamma_{\mathfrak{p}}(\xi),$$

for $\xi \in \mathfrak{k}$. In particular, the inclusion $U(\mathfrak{k}) \to U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$ is defined by the generators for the $\mathfrak{k}$-action.

PROOF. We have

$$j(\widehat{\xi}) = j(\tfrac{1}{2}\overline{\xi} - \gamma_{\mathfrak{k}}(\xi)) = \tfrac{1}{2}\overline{\xi} - \gamma_{\mathfrak{k}}(\xi) = \widehat{\xi} + \gamma_{\mathfrak{p}}(\xi). \qquad \square$$

Let $\phi_{\mathfrak{p}} \in \wedge^3\mathfrak{p}$ denote the cubic element, given as the projection of $\phi_{\mathfrak{g}} \in \wedge^3\mathfrak{g}$ relative to the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Fix a basis $\{e_a\}$ of $\mathfrak{g}$, given by a basis of $\mathfrak{k}$ followed by a basis of $\mathfrak{p}$.

LEMMA 4.7. *The decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *induces the following decomposition of the cubic element* $\phi_\mathfrak{g}$,

$$\phi_\mathfrak{g} = \phi_\mathfrak{k} + \phi_\mathfrak{p} + \sum\nolimits_a^{(\mathfrak{k})} \lambda_\mathfrak{p}(e_a) \wedge e^a.$$

Here $\sum_a^{(\mathfrak{k})}$ *indicates summation over the basis of* $\mathfrak{k}$.

PROOF. The difference $\phi_\mathfrak{g} - \phi_\mathfrak{k} - \phi_\mathfrak{p}$ are the 'mixed' components of $\phi_\mathfrak{g}$, i.e. those in $(\wedge^2\mathfrak{k}) \wedge \mathfrak{p} \oplus \mathfrak{k} \wedge (\wedge^2\mathfrak{p})$. Since $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, the element $\phi_\mathfrak{g}$ has no component in $(\wedge^2\mathfrak{k}) \wedge \mathfrak{p}$. The sum $\sum_a^{(k)} \lambda_\mathfrak{p}(e_a) \wedge e^a$ is the $\mathfrak{k} \wedge (\wedge^2\mathfrak{p})$-component, as one verifies by taking contractions with $\xi \in \mathfrak{k}$. $\square$

From the formula for $\phi_\mathfrak{g}$ (see (79)) we read off that

$$\phi_\mathfrak{p} = -\frac{1}{12} \sum\nolimits_{abc}^{(\mathfrak{p})} B([e_a, e_b]_\mathfrak{g}, e_c)\, e^a \wedge e^b \wedge e^c$$

where $\sum_{abc}^{(\mathfrak{p})}$ indicates a triple summation over the basis of $\mathfrak{p}$.

PROPOSITION 4.8. *The element* $\mathcal{D}(\mathfrak{g}, \mathfrak{k}) \in \mathcal{W}(\mathfrak{g}, \mathfrak{k}) = (U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p}))^{\mathfrak{k}-\mathrm{inv}}$ *is given by the formula*

$$\mathcal{D}(\mathfrak{g}, \mathfrak{k}) = \sum\nolimits_a^{(\mathfrak{p})} \widehat{e}_a e^a + q(\phi_\mathfrak{p}), \tag{89}$$

where $\sum_a^{(\mathfrak{p})}$ *indicates summation over the basis of* $\mathfrak{p}$.

PROOF. Using the formulas for $\mathcal{D}_\mathfrak{g}, \mathcal{D}_\mathfrak{k}$, and the property $j(\widehat{\xi}) = \widehat{\xi} + \gamma_\mathfrak{p}(\xi)$ we have

$$\mathcal{D}(\mathfrak{g}, \mathfrak{k}) = \sum\nolimits_a^{(\mathfrak{p})} \widehat{e}_a e^a + q\left(\phi_\mathfrak{g} - \phi_\mathfrak{k} - \sum\nolimits_a^{(\mathfrak{k})} \lambda_\mathfrak{p}(e_a) \wedge e^a\right).$$

By the Lemma, the expression in parentheses is $\phi_\mathfrak{p}$. $\square$

REMARK 4.9. The pair $\mathfrak{g}, \mathfrak{k}$ of quadratic Lie algebras is called a *symmetric pair* if $[\mathfrak{p}, \mathfrak{p}]_\mathfrak{g} \subset \mathfrak{k}$. Equivalently, $\phi_\mathfrak{p} = 0$. For this case, the element $\mathcal{D}(\mathfrak{g}, \mathfrak{k}) = \sum_a^{(\mathfrak{p})} \widehat{e}_a e^a$ was studied by Parthasarathy [**55**] in his 1972 paper *Dirac operator and the discrete series.* (Specifically, the context was that of a real semi-simple Lie group G with maximal compact subgroup K, with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition.) Kostant [**45**] discovered that for arbitrary pairs of quadratic Lie algebras, one obtains a Dirac operator with good properties if one adds the cubic term $q(\phi_\mathfrak{p})$. This motivated the terminology 'cubic Dirac operator'.

One can also define a morphism of the commutative Weil algebras $j\colon W\mathfrak{k} \to W\mathfrak{g}$, by the map $j(\xi) = \xi,\ j(\overline{\xi}) = \overline{\xi}$. The Poisson commutant of its image $j(W\mathfrak{k})$ is the $\mathfrak{k}$-basic subalgebra $W(\mathfrak{g}, \mathfrak{k}) = (W\mathfrak{g})_{\mathfrak{k}-\mathrm{bas}}$. Clearly, $W(\mathfrak{g}, \mathfrak{k})$ is the associated graded algebra to $\mathcal{W}(\mathfrak{g}, \mathfrak{k})$.

LEMMA 4.10. *We have commutative diagram of $\mathfrak{k}$-differential spaces,*

$$\begin{tikzcd} W\mathfrak{k} \arrow[r, "j"] \arrow[d, "q"] & W\mathfrak{g} \arrow[d, "q"] \\ \mathcal{W}\mathfrak{k} \arrow[r, "j"'] & \mathcal{W}\mathfrak{g} \end{tikzcd} \tag{90}$$

where the vertical maps are quantization maps for quantum Weil algebras, as defined in Section §7.3.

PROOF. Recall that the quantization maps are given by symmetrization in these variables $\xi, \overline{\xi}$. Since $j\colon \mathcal{W}\mathfrak{k} \to \mathcal{W}\mathfrak{g}$ and $j\colon W\mathfrak{k} \to W\mathfrak{g}$ are algebra homomorphisms, the symmetrization map commutes with j. □

The morphism of $\mathfrak{k}$-differential algebras $j\colon \mathcal{W}\mathfrak{k} \to \mathcal{W}\mathfrak{g}$ restricts to a morphism of $\mathfrak{k}$-basic subcomplexes. Since $(\mathcal{W}\mathfrak{k})_{\mathfrak{k}-\mathrm{bas}} = (U\mathfrak{k})^{\mathfrak{k}}$, while $(\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{bas}} = \mathcal{W}(\mathfrak{g}, \mathfrak{k})$, this gives a cochain map

$$j\colon (U\mathfrak{k})^{\mathfrak{k}} \to \mathcal{W}(\mathfrak{g}, \mathfrak{k}), \tag{91}$$

where $(U\mathfrak{k})^{\mathfrak{k}}$ carries the zero differential. Similarly, we obtain a cochain map $j\colon (S\mathfrak{k})^{\mathfrak{k}} \to W(\mathfrak{g}, \mathfrak{k})$.

PROPOSITION 4.11. *The maps in the commutative diagram*

$$\begin{tikzcd} (S\mathfrak{k})^{\mathfrak{k}} \arrow[r] \arrow[d] & H(W(\mathfrak{g}, \mathfrak{k})) \arrow[d] \\ (U\mathfrak{k})^{\mathfrak{k}} \arrow[r] & H(\mathcal{W}(\mathfrak{g}, \mathfrak{k})) \end{tikzcd} \tag{92}$$

obtained by taking the $\mathfrak{k}$-basic cohomology in (90)*, are all isomorphisms of algebras. In particular,*

$$H(\mathcal{W}(\mathfrak{g}, \mathfrak{k})) = (U\mathfrak{k})^{\mathfrak{k}}.$$

PROOF. We first show that the inclusion $j\colon W\mathfrak{k} \to W\mathfrak{g}$ is a $\mathfrak{k}$-homotopy equivalence, with homotopy inverse the projection $W\mathfrak{g} \to W\mathfrak{k}$. Let $W\mathfrak{p} = S(E_{\mathfrak{p}}[-1]) \subset W\mathfrak{g}$ be the subalgebra generated by $\xi \in \mathfrak{p}[-1]$, $\overline{\xi} \in \mathfrak{p}[-2]$ for $\xi \in \mathfrak{p}$. Note that $E_{\mathfrak{p}}[-1] \cong E_{\mathfrak{p}}[-1] \oplus 0$ is a $\mathfrak{k}$-differential subspace of $E_{\mathfrak{g}}[-1] \oplus \mathbb{K}\mathsf{c}$. Thus $W\mathfrak{g} = W\mathfrak{k} \otimes W\mathfrak{p}$ is an isomorphism of $\mathfrak{k}$-differential spaces. The homotopy equivalence between $S(E_{\mathfrak{p}}[-1])$ and $\mathbb{K}$ is compatible with the $\mathfrak{k}$-differential structure, i.e. it is a $\mathfrak{k}$-homotopy equivalence.

This gives the desired $\mathfrak{k}$-homotopy equivalence between $W\mathfrak{g}$ and $W\mathfrak{k}$. In particular, $j\colon W\mathfrak{k} \to W\mathfrak{g}$ induces an isomorphism in basic cohomology, proving that the upper horizontal map in (90) is an algebra isomorphism. Consequently the lower horizontal map is an algebra isomorphism as well. The left vertical map is an algebra isomorphism by Duflo's theorem. We conclude that the right vertical map is an algebra isomorphism. □

The inclusion

$$(U\mathfrak{g})^{\mathfrak{g}} = (\mathcal{W}\mathfrak{g})_{\mathfrak{g}-\mathrm{bas}} \hookrightarrow (\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{bas}} = \mathcal{W}(\mathfrak{g},\mathfrak{k})$$

of the $\mathfrak{g}$-basic subalgebra into $\mathfrak{k}$-basic subalgebra is a cochain map, defining an *algebra* homomorphism

$$(U\mathfrak{g})^{\mathfrak{g}} \to H(\mathcal{W}(\mathfrak{g},\mathfrak{k})) \cong (U\mathfrak{k})^{\mathfrak{k}}.$$

THEOREM 4.12. *The algebra homomorphism* $(U\mathfrak{g})^{\mathfrak{g}} \to (U\mathfrak{k})^{\mathfrak{k}}$ *fits into a commutative diagram,*

$$\begin{array}{ccc} (S\mathfrak{g})^{\mathfrak{g}} & \longrightarrow & (S\mathfrak{k})^{\mathfrak{k}} \\ \big\downarrow & & \big\downarrow \\ (U\mathfrak{g})^{\mathfrak{g}} & \longrightarrow & (U\mathfrak{k})^{\mathfrak{k}} \end{array}$$

where the vertical maps are Duflo isomorphisms, and the upper horizontal map is induced by the orthogonal projection $\mathrm{pr}_{\mathfrak{k}}\colon \mathfrak{g} \to \mathfrak{k}$.

PROOF. The result follows from the commutative diagram

$$\begin{array}{ccc} (W\mathfrak{g})_{\mathfrak{g}-\mathrm{bas}} & \longrightarrow & W(\mathfrak{g},\mathfrak{k}) \\ \big\downarrow & & \big\downarrow \\ (\mathcal{W}\mathfrak{g})_{\mathfrak{g}-\mathrm{bas}} & \longrightarrow & \mathcal{W}(\mathfrak{g},\mathfrak{k}) \end{array}$$

by passing to cohomology, using our results

$$H(W(\mathfrak{g},\mathfrak{k})) = (S\mathfrak{k})^{\mathfrak{k}},\quad H(\mathcal{W}(\mathfrak{g},\mathfrak{k})) = (U\mathfrak{k})^{\mathfrak{k}}.$$

The upper horizontal map $(S\mathfrak{g})^{\mathfrak{g}} \to (S\mathfrak{k})^{\mathfrak{k}}$ can be viewed as a composition

$$H_{\mathfrak{g}-\mathrm{bas}}(W\mathfrak{g}) \to H_{\mathfrak{k}-\mathrm{bas}}(W\mathfrak{g}) \to H_{\mathfrak{k}-\mathrm{bas}}(W\mathfrak{k}).$$

where the second map is induced by the morphism $W\mathfrak{g} \to W\mathfrak{k}$ given by projection of $\xi, \overline{\xi}$ to their $\mathfrak{k}$-components. This map takes $\lambda_{\mathfrak{g}}(\xi)$ for $\xi \in \mathfrak{g}$ to $\lambda_{\mathfrak{k}}(\mathrm{pr}_{\mathfrak{k}}(\xi))$. Hence, in terms of the variables $\xi, \widehat{\xi}$ it is still induced by the projection $\xi \mapsto \mathrm{pr}_{\mathfrak{k}}(\xi)$, $\widehat{\xi} \mapsto \widehat{\mathrm{pr}_{\mathfrak{k}}(\xi)}$. As a consequence, the map $(S\mathfrak{g})^{\mathfrak{g}} \to (S\mathfrak{k})^{\mathfrak{k}}$ is simply the map induced by the orthogonal projection. □

REMARK 4.13. Theorem 4.12 is a version of Vogan's conjecture (as formulated by Huang-Pandzic [**35**]) for quadratic Lie algebras. It was proved by Huang-Pandzic for symmetric pairs, and by Kostant [**47**] for reductive pairs. Kumar [**49**] interpreted Vogan's conjecture in terms of an induction map in the non-commutative equivariant cohomology from [**4**]. The simple proof given here, based on the quantization map for Weil algebras, is taken from [**5**].

5. Harish-Chandra projections

5.1. Harish-Chandra projections for enveloping algebras. A *triangular decomposition* of a Lie algebra $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{k} \oplus \mathfrak{n}_+$$

as vector spaces, where $\mathfrak{n}_-, \mathfrak{k}, \mathfrak{n}_+$ are Lie subalgebras of $\mathfrak{g}$ and $[\mathfrak{k}, \mathfrak{n}_\pm] \subset \mathfrak{n}_\pm$. The triangular decomposition determines characters

$$\kappa_\pm \in \mathfrak{k}^*, \quad \kappa_\pm(\xi) = \tfrac{1}{2} \operatorname{tr}_{\mathfrak{n}_\pm}(\mathrm{ad}_\xi).$$

By the Poincaré-Birkhoff-Witt theorem, the multiplication map

$$U(\mathfrak{n}_-) \otimes U(\mathfrak{k}) \otimes U(\mathfrak{n}_+) \to U(\mathfrak{g})$$

is an isomorphism of filtered vector spaces. By composing the inverse map with the projection to $U(\mathfrak{k})$ (using the augmentation maps $U(\mathfrak{n}_\pm) \to \mathbb{K}$), one obtains a map of filtered vector spaces

$$p_U \colon U(\mathfrak{g}) \to U(\mathfrak{k}), \tag{93}$$

left inverse to the inclusion $U(\mathfrak{k}) \hookrightarrow U(\mathfrak{g})$. We will refer to p_U as the *Harish-Chandra projection* for the given triangular decomposition of $\mathfrak{g}$. Equivalently, it is the projection to $U(\mathfrak{k})$ relative to the decomposition

$$U(\mathfrak{g}) = (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+) \oplus U(\mathfrak{k}).$$

EXAMPLE 5.1. Let $\mathfrak{g}$ be a complex reductive Lie algebra, and $\xi_0 \in \mathfrak{g}$ a semi-simple element, contained in some compact real form of $\mathfrak{g}$. Then the eigenvalues of $\mathrm{ad}(\xi_0)$ on $\mathfrak{g}$ are all purely imaginary. Let $\mathfrak{n}_-, \mathfrak{k}, \mathfrak{n}_+$ be the sum of eigenspaces of $\mathrm{ad}(\xi_0)$ of eigenvalues $2\pi\sqrt{-1}s$ with $s < 0, s = 0, s > 0$ respectively. Then $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{k} \oplus \mathfrak{n}_+$ is a triangular decomposition. If ξ_0 is a regular element so that $\mathfrak{k}$ is a Cartan subalgebra, the projection p_U is the classical Harish-Chandra projection. In this case, the character κ_+ coincides with $\rho \in \mathfrak{t}^*$, the half-sum of positive (complex) roots.

The projection (93) is not an algebra homomorphism, in general.

PROPOSITION 5.2. *The Harish-Chandra projection* (93) *intertwines the units, counits, comultiplications, and antipodes of the Hopf algebras* $U(\mathfrak{g})$ *and* $U(\mathfrak{k})$*. Suppose* $\mathfrak{s} \subset \mathfrak{k}$ *is a Lie subalgebra with the property*

$$(U(\mathfrak{k})U^+(\mathfrak{n}_+))^{\mathfrak{s}} = 0, \quad (U^+(\mathfrak{n}_-)U(\mathfrak{k}))^{\mathfrak{s}} = 0.$$

Then p_U *restricts to an algebra morphism*

$$U(\mathfrak{g})^{\mathfrak{s}} \to U(\mathfrak{k})^{\mathfrak{s}}$$

on $\mathfrak{s}$*-invariants. In particular, if such a subalgebra* $\mathfrak{s}$ *exists, then* p_U *is an algebra morphism on* $\mathfrak{g}$*-invariants.*

PROOF. It is obvious that p_U intertwines units and counits. The subspace $(\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+)$ is invariant under the antipode s of $U\mathfrak{g}$, hence p_U intertwines antipodes. Finally,

$$\begin{aligned}\ker(p_U) &= \Delta(\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+)\\ &\subset (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+) \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+)\\ &= \ker(p_U \otimes p_U)\end{aligned}$$

shows that it intertwines the comultiplications as well. The assumption on $\mathfrak{s}$ implies

$$(\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+)^{\mathfrak{s}} = (\mathfrak{n}_- U(\mathfrak{g})\mathfrak{n}_+)^{\mathfrak{s}},$$

which is an ideal in $U(\mathfrak{g})^{\mathfrak{s}}$. Hence the quotient map to $U(\mathfrak{k})^{\mathfrak{s}}$ is an algebra homomorphism. □

REMARK 5.3. The assumptions are satisfied in the setting of Example 5.1, by taking $\mathfrak{s} = \mathfrak{t}$ a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$. This follows by studying the decomposition into $\mathfrak{t}$-weight spaces.

Suppose that $\mathfrak{g}$ is a quadratic Lie algebra, with bilinear form B, and that the subalgebra $\mathfrak{k}$ is quadratic while $\mathfrak{n}_\pm$ are isotropic. We will assume furthermore that $\mathfrak{g}^{\mathfrak{k}} \subset \mathfrak{k}$, which is thus the center of $\mathfrak{k}$. Since B defines a $\mathfrak{k}$-equivariant non-degenerate pairing between $\mathfrak{n}_\pm$, we have $\kappa_+ = -\kappa_-$. Put $\kappa = \kappa_+$.

Letting e_α be a basis of $\mathfrak{n}_+$, and e^α the B-dual basis of $\mathfrak{n}_- \cong (\mathfrak{n}_+)^*$, the element

$$\sum_\alpha e_\alpha \otimes e^\alpha \in \mathfrak{n}_+ \otimes \mathfrak{n}_-$$

does not depend on the choice of basis, and is $\mathfrak{k}$-invariant. (Under the identification $\mathfrak{n}_+ \otimes \mathfrak{n}_- = \mathfrak{n}_+ \otimes \mathfrak{n}_- = \mathrm{End}(\mathfrak{n}_+)$, it corresponds to the identity endomorphism.) Hence, the element

$$\kappa^\sharp = \tfrac{1}{2} \sum_\alpha [e_\alpha, e^\alpha]_{\mathfrak{g}}$$

is $\mathfrak{k}$-invariant and therefore lies in the center of $\mathfrak{k}$.

Let $\mathrm{Cas}_{\mathfrak{g}} \in U(\mathfrak{g})$ be the quadratic Casimir defined by B, and $\mathrm{Cas}_{\mathfrak{k}}$ the Casimir defined by the restriction of B to $\mathfrak{k}$.

PROPOSITION 5.4. *The Harish-Chandra projection of the quadratic Casimir element is given by*

$$p_U(\mathrm{Cas}_{\mathfrak{g}}) = \mathrm{Cas}_{\mathfrak{k}} + 2\kappa^\sharp.$$

PROOF. Let e_i be a basis of $\mathfrak{k}$, with B-dual basis e^i, and let e_α be a basis of $\mathfrak{n}_+$, with B-dual basis e^α of $\mathfrak{n}_-$. Then

$$\begin{aligned}\mathrm{Cas}_{\mathfrak{g}} &= \sum_i e_i e^i + \sum_\alpha (e_\alpha e^\alpha + e^\alpha e_\alpha)\\ &= \mathrm{Cas}_{\mathfrak{k}} + 2\sum_\alpha e^\alpha e_\alpha + 2\kappa^\sharp\end{aligned}$$

Harish-Chandra projection removes the second term. □

REMARK 5.5. Given a central extension $0 \to \mathbb{K}\mathsf{c} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$, the triangular decomposition of $\mathfrak{g}$ induces a triangular decomposition of $\widehat{\mathfrak{g}}$. For any $r \in \mathbb{K}$ one may thus define Harish-Chandra projections

$$p_U \colon U_r(\widehat{\mathfrak{g}}) \to U_r(\widehat{\mathfrak{k}})$$

for the level r enveloping algebras.

REMARK 5.6. The notion of triangular decomposition, with the corresponding Harish-Chandra projection, generalizes with obvious changes to the case of super Lie algebras.

5.2. Harish-Chandra projections for Clifford algebras. Given a quadratic vector space (V, B) and a decomposition $V = V_- \oplus V_0 \oplus V_+$, where V_0 is a quadratic subspace and $V_0^\perp = V_+ \oplus V_-$ a splitting of its orthogonal by two isotropic subspaces, one has an isomorphism

$$\mathrm{Cl}(V) \cong \wedge(V_-) \otimes \mathrm{Cl}(V_0) \otimes \wedge(V_+),$$

as algebras, and the augmentation maps for $\wedge V_\pm$ define a Harish-Chandra projection

$$p_{\mathrm{Cl}} \colon \mathrm{Cl}(V) \to \mathrm{Cl}(V_0).$$

Equivalently, this is the projection along $V_-\mathrm{Cl}(V) + \mathrm{Cl}(V)V_+$.

REMARK 5.7. Thinking of $\mathrm{Cl}(V)$ as the level 1 enveloping algebra of the filtered super Lie algebra $\mathbb{K} \oplus V[-1]$ (cf. §5.1.9) this is may be viewed as a Harish-Chandra projection

$$U_1(\mathbb{K} \oplus V[-1]) \to U_1(\mathbb{K} \oplus V_0[-1]),$$

as in Remarks 5.5, 5.6.

We are interested in the setting from Section 5.1, where $V = \mathfrak{g}$ is a quadratic Lie algebra, and $V_0 = \mathfrak{k}$, $V_\pm = \mathfrak{n}_\pm$ the summands of a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{k} \oplus \mathfrak{n}_+$.

LEMMA 5.8. *The Harish-Chandra projection* $p_{\mathrm{Cl}} \colon \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{k})$ *intertwines contractions* $\iota(\xi)$ *and Lie derivatives* $L(\xi)$ *for* $\xi \in \mathfrak{k}$.

PROOF. The subspace $\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g}) + \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$ is invariant under $\iota(\xi),\ L(\xi)$. □

The projection γ_{Cl} does not intertwine the Clifford differentials in general: See Proposition 5.12 below.

Let $\kappa \in \mathfrak{k}^*$ as above. Let $\phi_\mathfrak{g} \in \wedge^3\mathfrak{g}$, $\phi_\mathfrak{k} \in \wedge^3\mathfrak{k}$ be the structure constants tensors (cf. §7.1).

PROPOSITION 5.9. *The Harish-Chandra projection of the quantized structure constants tensor is given by*

$$p_{\mathrm{Cl}}(q(\phi_\mathfrak{g})) = q(\phi_\mathfrak{k}) + \kappa^\sharp.$$

For any $\xi \in \mathfrak{k}$, *the Harish-Chandra projection of* $\gamma_\mathfrak{g}(\xi)$ *is*

$$p_{\mathrm{Cl}}(\gamma_\mathfrak{g}(\xi)) = \gamma_\mathfrak{k}(\xi) + \kappa(\xi).$$

PROOF. We use the basis from the proof of Proposition 5.4. We have the formula,

$$\gamma_{\mathfrak{g}}(\xi) = \frac{1}{4}\sum_i [\xi, e_i]e^i + \frac{1}{4}\sum_\alpha [\xi, e^\alpha]_{\mathfrak{g}} e_\alpha + \frac{1}{4}\sum_\alpha [\xi, e_\alpha]_{\mathfrak{g}} e^\alpha \tag{94}$$

The first term is $\gamma_{\mathfrak{k}}(\xi)$. The second term vanishes under p_{Cl}. The last term has Harish-Chandra projection

$$\frac{1}{2}\sum_\alpha B([\xi, e_\alpha]_{\mathfrak{g}}, e^\alpha) = \kappa(\xi).$$

It follows from parity considerations that $p_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}})) - q(\phi_{\mathfrak{k}}) \in \mathfrak{k}$. To compute this element we apply $\iota(\xi)$. We have

$$\iota(\xi)(p_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}})) - q(\phi_{\mathfrak{k}})) = p_{\mathrm{Cl}}(\iota(\xi)q(\phi_{\mathfrak{g}})) - \iota(\xi)q(\phi_{\mathfrak{k}}) = p_{\mathrm{Cl}}(\gamma_{\mathfrak{g}}(\xi)) - \gamma_{\mathfrak{k}}(\xi) = \kappa(\xi).$$

This shows $p_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}})) - q(\phi_{\mathfrak{k}}) = \kappa^\sharp$. □

PROPOSITION 5.10. *Let $\Gamma \in \mathrm{Cl}(\mathfrak{g})$ be a chirality element for $\mathfrak{g}$ (i.e. the quantization of a generator $\Gamma_\wedge$ of $\det(\mathfrak{g})$). Then $p_{\mathrm{Cl}}(\Gamma)$ is a chirality element of $\mathrm{Cl}(\mathfrak{k})$.*

PROOF. Let e_i be a basis of $\mathfrak{k}$, and e_α, e^α the B-dual bases of $\mathfrak{n}_\pm$. We may take Γ to be the quantization of the wedge product of all these basis vectors.

$$\begin{aligned}
\Gamma &= q\left((\prod_\alpha e^\alpha \wedge e_\alpha)(\prod_i e_i) \right) \\
&= \prod_\alpha q(e^\alpha \wedge e_\alpha)\ q(\prod_i e_i) \\
&= \prod_\alpha (e^\alpha e_\alpha - 1)\ q(\prod_i e_i).
\end{aligned}$$

We read off that the Harish-Chandra projection is $\pm q(\prod_i e_i)$. If Γ is normalized so that $\Gamma^2 = 1$, then $p_{\mathrm{Cl}}(\Gamma)^2 = 1$ (since p_{Cl} is an algebra morphism on invariants). □

The Lie algebra homomorphism $\gamma_{\mathfrak{g}} \colon \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ extends to an algebra homomorphism $\gamma_{\mathfrak{g}} \colon U(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$, and similarly for $\mathfrak{k}$. These homomorphisms do not intertwine the Harish-Chandra projections, in general. However, one has the following slightly weaker statements. Let $\tau \colon U(\mathfrak{k}) \to U(\mathfrak{k})$ be the automorphism of $U(\mathfrak{k})$ extending the map $\mathfrak{k} \to U(\mathfrak{k}),\ \xi \mapsto \xi + \kappa(\xi)$.

PROPOSITION 5.11. *The following diagram commutes:*

$$\begin{tikzcd}
U(\mathfrak{g}) \arrow[r, "\gamma_{\mathfrak{g}}"'] \arrow[d, "\tau\circ p_U"] & \mathrm{Cl}(\mathfrak{g}) \arrow[d, "p_{\mathrm{Cl}}"] \\
U(\mathfrak{k}) \arrow[r, "\gamma_{\mathfrak{k}}"'] & \mathrm{Cl}(\mathfrak{k})
\end{tikzcd}$$

Proof. From (94) we deduce that

$$\gamma_{\mathfrak{g}}(\xi) = \gamma_{\mathfrak{k}}(\xi) + \kappa(\xi) \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+ = \gamma_{\mathfrak{k}}(\tau(\xi)) \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$$

if $\xi \in \mathfrak{k}$, while $\gamma_{\mathfrak{g}}(\xi) \in \mathfrak{n}_-U(\mathfrak{g})\mathfrak{n}_+$ for $\xi \in \mathfrak{n}_\pm$. Since $\mathfrak{n}_-U(\mathfrak{g})\mathfrak{n}_+$ is a 2-sided ideal in $U(\mathfrak{g})$, this implies

$$\gamma_{\mathfrak{g}}(\xi_1 \cdots \xi_r) = \gamma_{\mathfrak{k}}(\tau(\xi_1)) \cdots \gamma_{\mathfrak{k}}(\tau(\xi_r)) \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$$

for all $\xi_1, \ldots, \xi_r \in \mathfrak{g}$. □

Proposition 5.12. *The Harish-Chandra projection p_{Cl} intertwines the Clifford differential d on $\mathrm{Cl}(\mathfrak{g})$ with the differential $d + 2\iota(\kappa)$ on $\mathrm{Cl}(\mathfrak{k})$.*

Proof. We denote the two Clifford differentials by $\mathrm{d}_{\mathfrak{g}}$, $\mathrm{d}_{\mathfrak{k}}$ for clarity. Recall that $\mathrm{d}_{\mathfrak{g}}\xi = 2\gamma(\xi), \quad \xi \in \mathfrak{g}$. As observed in the proof of Proposition 5.11, we have $\gamma_{\mathfrak{g}}(\mathfrak{n}_-) \in \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})$ and $\gamma_{\mathfrak{g}}(\mathfrak{n}_+) \in \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$. It follows that $\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})$ and $\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$ are differential subspaces of $\mathrm{Cl}(\mathfrak{g})$. On the other hand, for $\xi \in \mathfrak{k} \subset \mathrm{Cl}(\mathfrak{g})$ we have

$$\begin{aligned} \mathrm{d}_{\mathfrak{g}}\xi &= 2\gamma_{\mathfrak{g}}(\xi) \\ &= 2(\gamma_{\mathfrak{k}}(\xi) + \kappa(\xi)) \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+ \\ &= (\mathrm{d}_{\mathfrak{k}} + 2\iota(\kappa))\xi \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+. \end{aligned}$$

Hence, if $x \in \mathrm{Cl}(\mathfrak{k}) \subset \mathrm{Cl}(\mathfrak{g})$, $\mathrm{d}_{\mathfrak{g}}x = (\mathrm{d}_{\mathfrak{k}} + 2\iota(\kappa))x \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$. □

Proposition 5.13. *Suppose $\mathfrak{s} \subset \mathfrak{g}$ preserves $\mathfrak{k}, \mathfrak{n}_+, \mathfrak{n}_-$ and that*

$$(\wedge^+(\mathfrak{n}_-)\mathrm{Cl}(\mathfrak{k}))^{\mathfrak{s}} = 0, \ (\mathrm{Cl}(\mathfrak{k}) \wedge^+ (\mathfrak{n}_+))^{\mathfrak{s}} = 0.$$

Then the Harish-Chandra projection γ_{Cl} restricts to an algebra morphism on $\mathfrak{s}$-invariants. In particular, if such a subalgebra $\mathfrak{s}$ exists, then γ_{Cl} restricts to an algebra morphism $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cl}(\mathfrak{k})$. Furthermore, in this case

$$\frac{1}{12}\,\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}})) - \frac{1}{12}\,\mathrm{tr}_{\mathfrak{k}}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{k}})) = 2\langle \kappa, \kappa^\sharp \rangle. \tag{95}$$

Proof. The conditions on $\mathfrak{s}$ imply

$$\mathrm{Cl}(\mathfrak{g})^{\mathfrak{s}} = (\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+)^{\mathfrak{s}} \oplus \mathrm{Cl}(\mathfrak{k})^{\mathfrak{s}}.$$

Since $(\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+)^{\mathfrak{s}}$ is a 2-sided ideal in $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{s}}$, the quotient map $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{s}} \to \mathrm{Cl}(\mathfrak{k})^{\mathfrak{s}}$ is an algebra homomorphism. Let us use this together with the formula $p_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}})) = q(\phi_{\mathfrak{k}}) + \kappa^\sharp$ to compute $[p_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}})), p_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}}))]$. We have $[q(\phi_{\mathfrak{k}}), \kappa^\sharp] = 2\gamma_{\mathfrak{k}}(\kappa^\sharp) = 0$ since $\kappa^\sharp$ is central in $\mathfrak{k}$. Hence

$$[p_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}})), p_{\mathrm{Cl}}(q(\phi_{\mathfrak{g}}))] = [q(\phi_{\mathfrak{k}}), q(\phi_{\mathfrak{k}})] + 2B(\kappa^\sharp, \kappa^\sharp).$$

On the other hand, we have $[q(\phi_{\mathfrak{k}}), q(\phi_{\mathfrak{k}})] = \frac{1}{12}\,\mathrm{tr}_{\mathfrak{k}}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{k}}))$ and $[q(\phi_{\mathfrak{g}}), q(\phi_{\mathfrak{g}})] = \frac{1}{12}\,\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}}))$, cf. §7.1. □

The assumptions of the Proposition hold for the standard example 5.1, by taking $\mathfrak{s} = \mathfrak{t}$ to be a Cartan subalgebra in $\mathfrak{k}$. In this case, the Proposition above is a version of the *Freudenthal-de Vries strange formula.* (See [**45**, Proposition 1.84].)

5.3. Harish-Chandra projections for quantum Weil algebras. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{k} \oplus \mathfrak{n}_+$ be as in the last Section. Let $\mathcal{W}(\mathfrak{g})$ be the corresponding quantum Weil algebra. Note that the filtered Lie algebra $\widetilde{\mathfrak{n}_+} = \mathfrak{n}_+[-1] \rtimes \mathfrak{n}_+[-2]$ (spanned by $\xi, \overline{\xi}$ for $\xi \in \mathfrak{n}_+$) is a $\mathfrak{k}$-differential subspace of $\mathcal{W}\mathfrak{g}$, and similarly for $\widetilde{\mathfrak{n}_-} = \mathfrak{n}_-[-1] \rtimes \mathfrak{n}_-[-2]$. Hence $\widetilde{\mathfrak{n}_-}\mathcal{W}(\mathfrak{g}) + \mathcal{W}(\mathfrak{g})\widetilde{\mathfrak{n}_+}$ is a $\mathfrak{k}$-differential subspace. It defines a complement to $\mathcal{W}(\mathfrak{k})$, and projection onto the second summand in

$$\mathcal{W}(\mathfrak{g}) = (\widetilde{\mathfrak{n}_-}\mathcal{W}(\mathfrak{g}) + \mathcal{W}(\mathfrak{g})\widetilde{\mathfrak{n}_+}) \oplus \mathcal{W}(\mathfrak{k})$$

will be called the Harish-Chandra projection

$$p_{\mathcal{W}} \colon \mathcal{W}(\mathfrak{g}) \to \mathcal{W}(\mathfrak{k})$$

for the quantum Weil algebras.

Put differently, $p_{\mathcal{W}}$ is obtained by composing the inverse of the map

$$U(\widetilde{\mathfrak{n}_-}) \otimes \mathcal{W}(\mathfrak{k}) \otimes U(\widetilde{\mathfrak{n}_+}) \to \mathcal{W}(\mathfrak{g})$$

with the augmentation maps for $U(\widetilde{\mathfrak{n}_\pm})$.

PROPOSITION 5.14. *The Harish-Chandra projection $p_{\mathcal{W}}$ is a morphism of $\mathfrak{k}$-differential spaces, left inverse to the inclusion $j \colon \mathcal{W}\mathfrak{k} \hookrightarrow \mathcal{W}\mathfrak{g}$. It restricts to a cochain map $\mathcal{W}(\mathfrak{g}, \mathfrak{k}) \to U(\mathfrak{k})^{\mathfrak{k}}$.*

PROOF. The first part is evident from the construction. The second part follows since $p_{\mathcal{W}}$ takes $\mathcal{W}(\mathfrak{g})_{\mathfrak{k}-\mathrm{bas}} = \mathcal{W}(\mathfrak{g}, \mathfrak{k})$ to $\mathcal{W}(\mathfrak{k})_{\mathfrak{k}-\mathrm{bas}} = (U\mathfrak{k})^{\mathfrak{k}}$. $\square$

PROPOSITION 5.15. *The Harish-Chandra projection of the cubic Dirac operator for $\mathfrak{g}$*

$$p_{\mathcal{W}}(\mathcal{D}_{\mathfrak{g}}) = \mathcal{D}_{\mathfrak{k}},$$

the cubic Dirac operator for $\mathfrak{k}$.

PROOF. Recall that $\mathcal{D}_{\mathfrak{g}} = j(\mathcal{D}_{\mathfrak{k}}) + \mathcal{D}(\mathfrak{g}, \mathfrak{k})$. The Harish-Chandra projection $p_{\mathcal{W}}(\mathcal{D}(\mathfrak{g}, \mathfrak{k}))$ vanishes since $j(\mathcal{W}(\mathfrak{g}, \mathfrak{k}))$ consists of even elements. $\square$

For the commutative Weil algebras, one has a much simpler projection

$$p_W \colon W(\mathfrak{g}) \to W(\mathfrak{k})$$

induced by the projection $\mathfrak{g} \to \mathfrak{k}$.

THEOREM 5.16. *The following diagram commutes up to $\mathfrak{k}$-homotopy:*

$$\begin{array}{ccc} W(\mathfrak{g}) & \xrightarrow[q]{} & \mathcal{W}(\mathfrak{g}) \\ \Big\downarrow{\scriptstyle p_W} & & \Big\downarrow{\scriptstyle p_{\mathcal{W}}} \\ W(\mathfrak{k}) & \xrightarrow[q]{} & \mathcal{W}(\mathfrak{k}) \end{array}$$

It gives rise to a commutative diagram,

$$\begin{tikzcd} S(\mathfrak{g})^{\mathfrak{g}} \arrow[r, "q"'] \arrow[d, "p_S"] & U(\mathfrak{g})^{\mathfrak{g}} \arrow[d, "\tau\circ p_U"] \\ S(\mathfrak{k})^{\mathfrak{k}} \arrow[r, "q"'] & U(\mathfrak{k})^{\mathfrak{k}} \end{tikzcd}$$

where the horizontal maps are Duflo maps.

Proof. Since $\widetilde{\mathfrak{n}_\pm}$ are $\mathfrak{k}$-differential spaces, the augmentation maps

$$U(\widetilde{\mathfrak{n}_\pm}) \to \mathbb{K}$$

are $\mathfrak{k}$-homotopy equivalences. $\square$

CHAPTER 8

Applications to reductive Lie algebras

We will now apply the results above to the case that $\mathfrak{g}$ is a complex reductive Lie algebra.

1. Notation

We refer to Appendix B for background information on reductive Lie algebras. Let us list some of the notation introduced there.

Fix a compact real form $\mathfrak{g}_{\mathbb{R}}$, and let $\xi \to \xi^*$ be the corresponding complex conjugation mapping for $\mathfrak{g}$. We assume that B is an invariant non-degenerate symmetric $\mathbb{C}$-bilinear on $\mathfrak{g}$, given as the complexification of an $\mathbb{R}$-bilinear form on $\mathfrak{g}_{\mathbb{R}}$. Sine B is non-degenerate, it identifies $\mathfrak{g}$ and $\mathfrak{g}^*$; the resulting bilinear form on $\mathfrak{g}^*$ will be denoted by B^*.

We denote by $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra, obtained by complexifying a maximal abelian subalgebra $\mathfrak{t}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$. Let $\mathfrak{R} \subset \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^*$ be the set of roots of $\mathfrak{g}$, and $\mathfrak{g}_\alpha \subset \mathfrak{g}$ the root space corresponding to the root $\alpha \in \mathfrak{R}$. Fix a positive Weyl chamber $\mathfrak{t}_+$, corresponding to a decomposition $\mathfrak{R} = \mathfrak{R}_+ \cup \mathfrak{R}_-$ into positive and negative roots. We let $e_\alpha \in \mathfrak{g}$ for $\alpha \in \mathfrak{R}$ be root vectors, normalized in such a way that $e_\alpha^* = e_{-\alpha}$ and $B(e_\alpha, e_{-\alpha}) = 1$.

The weight lattice will be denoted P, and the set of dominant weights P_+. Then P_+ labels the irreducible finite-dimensional representations of $\mathfrak{g}$. We let $V(\mu)$ denote the irreducible $\mathfrak{g}$-representation of highest weight $\mu \in P_+$.

2. Harish-Chandra projections

Consider the nilpotent Lie subalgebras

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in \mathfrak{R}_\pm} \mathfrak{g}_\alpha,$$

with basis the root vectors e_α for $\alpha \in \mathfrak{R}_\pm$. Then $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$ is the standard triangular decomposition of $\mathfrak{g}$. The Harish-Chandra projection (§7, Section 5.1)

$$p_U \colon U(\mathfrak{g}) \to U(\mathfrak{t}) = S(\mathfrak{t})$$

defined by this decomposition restricts to an algebra homomorphism on $\mathfrak{g}$-invariants (cf. §7.5.2). We have, for all $\xi \in \mathfrak{t}$,

$$\mathrm{tr}_{\mathfrak{n}_+}(\mathrm{ad}(\xi)) = \sum_{\alpha \in \mathfrak{R}_+} \langle \alpha, \xi \rangle = 2\langle \rho, \xi \rangle$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\mathfrak{R}_+}\alpha \in \mathfrak{t}^*$ is the half-sum of positive roots of $\mathfrak{g}$. Consequently

$$p_U(\mathrm{Cas}_\mathfrak{g}) = \mathrm{Cas}_\mathfrak{t} + 2B^\sharp(\rho)$$

(cf. §7.5.4). This has the following well-known consequence.

PROPOSITION 2.1. *The action of the quadratic Casimir element* $\mathrm{Cas}_\mathfrak{g}$ *in the irreducible unitary representation* $\pi\colon \mathfrak{g} \to \mathrm{End}(V)$ *of highest weight* $\mu \in P_+ \subset \mathfrak{t}^*$ *is given by the scalar,*

$$\pi(\mathrm{Cas}_\mathfrak{g}) = B^*(\mu+\rho, \mu+\rho) - B^*(\rho, \rho).$$

PROOF. Since V is irreducible, $\mathrm{Cas}_\mathfrak{g}$ acts as a scalar. To find this scalar, evaluate on the highest weight vector $v \in V$. $\mathrm{Cas}_\mathfrak{g}$ decomposes into two summands according to $U(\mathfrak{g})^\mathfrak{t} = (\mathfrak{n}_- U(\mathfrak{g})\mathfrak{n}_+)^\mathfrak{t} \oplus U(\mathfrak{t})$. The first summand acts trivially on v. The second summand is the Harish-Chandra projection $\mathrm{Cas}_\mathfrak{t} + 2B^\sharp(\rho)$, and it acts on v as $B^*(\mu,\mu) + 2B^*(\rho,\mu) = B^*(\mu+\rho,\mu+\rho) - B^*(\rho,\rho)$. □

REMARK 2.2. Note that if the bilinear form B is *positive definite* on $\mathfrak{g}_\mathbb{R}$, then B^* is negative definite on the real subspace space spanned by the weights. In this case, the right hand side can be written $-||\mu+\rho||^2 + ||\rho||^2$.

As a special case, suppose $\mathfrak{g}$ is simple, and take V to be the adjoint representation. The highest weight of this representation is, by definition, the highest root $\alpha_{\max}$ of $\mathfrak{g}$. Let $\alpha_{\max}^\vee$ be the corresponding co-root (cf. Definition 4.3 in Appendix B), and

$$h^\vee = 1 + \langle \rho,\ \alpha_{\max}^\vee\rangle$$

the *dual Coxeter number.* The *basic inner product* on a simple Lie algebra $\mathfrak{g}$ is the unique invariant inner product such that $B_{\mathrm{basic}}(\alpha_{\max}^\vee, \alpha_{\max}^\vee) = 2$. Note that B_{basic} is *negative definite* on $\mathfrak{g}_\mathbb{R}$.

PROPOSITION 2.3. *If* $\mathfrak{g}$ *is simple, and* $B = B_{\mathrm{basic}}$ *is the basic inner product on* $\mathfrak{g}$*, the adjoint action of the quadratic Casimir is given by the scalar*

$$\mathrm{ad}(\mathrm{Cas}_\mathfrak{g}) = 2h^\vee.$$

PROOF. By §8, Proposition 2.1, $\mathrm{ad}(\mathrm{Cas}_\mathfrak{g})$ is the scalar,

$$B^*(\alpha_{\max}, \alpha_{\max}) + 2B^*(\alpha_{\max}, \rho) = B^*(\alpha_{\max}, \alpha_{\max})\ h^\vee = 2h^\vee. \qquad \square$$

PROPOSITION 2.4. *The basic inner product* B_{basic} *is related to the Killing form* $B_{\mathrm{Killing}}(\xi,\xi') = \mathrm{tr}(\mathrm{ad}_\xi\, \mathrm{ad}_{\xi'})$*,* $\xi, \xi' \in \mathfrak{g}$ *by twice the dual Coxeter number:*

$$B_{\mathrm{Killing}} = 2h^\vee B_{\mathrm{basic}}.$$

PROOF. Let $\mathrm{Cas}'_\mathfrak{g}$ be the Casimir operator relative to $B' = B_{\mathrm{Killing}}$. Since $\mathfrak{g}$ is simple, we have $B_{\mathrm{Killing}} = tB_{\mathrm{basic}}$ for soome $t \neq 0$, and hence $\mathrm{Cas}'_\mathfrak{g} = \frac{1}{t}\mathrm{Cas}_\mathfrak{g}$. By definition of the Killing form, the trace $\mathrm{ad}(\mathrm{Cas}'_\mathfrak{g})$ equals $\dim\mathfrak{g}$. This shows that $\mathrm{ad}(\mathrm{Cas}'_\mathfrak{g})$ acts as 1 in the adjoint representation. Comparing with Proposition 2.3, it follows that $\frac{1}{t} = 2h^\vee$. □

The Harish-Chandra projection for the Clifford algebras $p_{\mathrm{Cl}}\colon \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{t})$ (§7, Section 5.2) has interesting consequences as well. By §7, Proposition 5.9 we have

$$p_{Cl}(q(\phi)) = B^\sharp(\rho).$$

§7, Proposition 5.13 specializes to

PROPOSITION 2.5 (Freudenthal-de Vries). *The length squared of the element ρ is given by*

$$B^*(\rho,\rho) = \frac{1}{24}\operatorname{tr}(\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}})).$$

REMARK 2.6. The above version of the Freudenthal-de Vries formula, valid for arbitrary bilinear forms, was formulated by Kostant. If $\mathfrak{g}$ is semisimple and $B = B_{\mathrm{Killing}}$, so that $\mathrm{ad}(\mathrm{Cas}_{\mathfrak{g}}) = 1$, one obtains the more standard version of the formula,

$$B^*_{\mathrm{Killing}}(\rho,\rho) = \frac{\dim\mathfrak{g}}{24}.$$

3. The ρ-representation and the representation theory of $\mathrm{Cl}(\mathfrak{g})$

The irreducible representation $\pi\colon \mathfrak{g} \to \mathrm{End}(V(\rho))$ of highest weight ρ is closely related to Clifford algebras. Many of the results in this Section are due to Kostant [**44**]. However, our proofs are rather different. We begin with the following simple observation.

PROPOSITION 3.1. *Let E be a finite-dimensional $\mathrm{Cl}(\mathfrak{g})$-module. Let $\mathfrak{g}$ act on E via the map $\gamma\colon \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$. Then E is a direct sum of ρ-representations.*

PROOF. For $\xi \in \mathfrak{t}$, we have

$$\gamma(\xi) = \langle\rho,\xi\rangle \mod \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+. \tag{96}$$

This shows that ξ acts on highest weight vectors as a scalar $\langle\rho,\xi\rangle$. □

Thus, if E is a finite-dimensional $\mathrm{Cl}(\mathfrak{g})$-module, then any highest weight vector $v \in E^{\mathfrak{n}_+}$ generates an irreducible representation $U(\mathfrak{g}).v$ isomorphic to $V(\rho)$. To obtain a concrete model, take $E = \mathrm{Cl}(\mathfrak{g})$ with the left regular representation of $\mathrm{Cl}(\mathfrak{g})$ on itself. The line $\det(\mathfrak{n}_+)\det(\mathfrak{n}_-) \subset \mathrm{Cl}(\mathfrak{g})^{\mathfrak{t}}$ has a unique generator

$$\mathsf{R} \in \det(\mathfrak{n}_+)\det(\mathfrak{n}_-)$$

with the property $\mathsf{R}^2 = \mathsf{R}$. Equivalently, R is normalized by the property $p_{\mathrm{Cl}}(\mathsf{R}) = 1$. In terms of (normalized) root vectors,

$$\mathsf{R} = \prod_{\alpha\in\mathfrak{R}_+} \tfrac{1}{2}e_\alpha e_{-\alpha} \in \mathrm{Cl}(\mathfrak{g}). \tag{97}$$

PROPOSITION 3.2. *There is an isomorphism of $\mathfrak{g}$-representations,*

$$V(\rho) \cong \gamma(U(\mathfrak{g}))\mathsf{R}. \tag{98}$$

PROOF. Since $\gamma(\mathfrak{n}_+) \subset \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$, it is immediate that R is a highest weight vector. □

We can also consider $\mathrm{Cl}(\mathfrak{g})$ as a $\mathfrak{g} \times \mathfrak{g}$-representation, by composing the homomorphism $\gamma\colon \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ with the left and right regular representations of $\mathrm{Cl}(\mathfrak{g})$ on itself. That is,

$$(\xi_1, \xi_2).x = \gamma(\xi_1)x - x\gamma(\xi_2)$$

for $\xi_1, \xi_2 \in \mathfrak{g}$. Take the Weyl chamber for $\mathfrak{g}\times\mathfrak{g}$ to be the product $\mathfrak{t}_+ \times (-\mathfrak{t}_+)$. The set of positive roots for $\mathfrak{g} \times \mathfrak{g}$ is then

$$\mathfrak{R}_+ \times \{0\} \cup \{0\} \times \mathfrak{R}_-,$$

and the space of highest weight vectors for a $\mathfrak{g} \times \mathfrak{g}$-representation is the subspace fixed by $\mathfrak{n}_+ \times \mathfrak{n}_-$. Proposition 3.1 shows that the highest weights for the $\mathfrak{g} \times \mathfrak{g}$-action on $\mathrm{Cl}(\mathfrak{g})$ are all equal to $(\rho, -\rho)$. The same is true for all $\mathfrak{g} \times \mathfrak{g}$-subrepresentations, in particular for $\gamma(U(\mathfrak{g})) \subset \mathrm{Cl}(\mathfrak{g})$.

Given a finite-dimensional $\mathfrak{g}$-representation V, let $\mathrm{End}(V)$ carry the $\mathfrak{g}\times\mathfrak{g}$-representation obtained by composing $\pi\colon \mathfrak{g} \to \mathrm{End}(V)$ with the left and right regular representations of $\mathrm{End}(V)$ on itself:

$$(\xi_1, \xi_2).A = \pi(\xi_1)A - A\pi(\xi_2).$$

If $V(\mu)$ is an irreducible unitary $\mathfrak{g}$-representation of highest weight μ, then $\mathrm{End}(V(\mu))$ is an irreducible unitary $\mathfrak{g} \times \mathfrak{g}$-representation of highest weight $(\mu, -\mu)$, and with highest weight vector the orthogonal projection $\mathrm{pr}_{V(\mu)^{\mathfrak{n}_+}}$ onto $V(\mu)^{\mathfrak{n}_+}$.

Proposition 3.3. *The action of $\gamma(U(\mathfrak{g}))$ by left multiplication on $V(\rho) \cong \gamma(U(\mathfrak{g}))\mathsf{R}$ defines an algebra isomorphism*

$$\mathrm{End}(V(\rho)) \cong \gamma(U(\mathfrak{g})).$$

The space $\gamma(U(\mathfrak{g}))^{\mathfrak{n}_+\times\mathfrak{n}_-}$ of highest weight vectors is spanned by the element R.

Proof. The element R is a highest weight vector for the $\mathfrak{g}\times\mathfrak{g}$-action on $\mathrm{Cl}(\mathfrak{g})$, of weight $(\rho, -\rho)$. Hence $\gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g})) \subset \mathrm{Cl}(g)$ is an irreducible $\mathfrak{g} \times \mathfrak{g}$-representation of highest weight $(\rho, -\rho)$. Using identities such as

$$\tfrac{1}{2}e_{-\alpha}(\tfrac{1}{2}e_\alpha e_{-\alpha})e_\alpha + \tfrac{1}{2}e_\alpha e_{-\alpha} = 1,$$

one sees that $1 \in \gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g}))$, hence $\gamma(U(\mathfrak{g})) \subset \gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g}))$. Since the right hand side is an irreducible $\mathfrak{g} \times \mathfrak{g}$-representation, this inclusion is an equality. In particular, $\gamma(U(\mathfrak{g}))$ is irreducible, with $\mathsf{R} \in \gamma(U(\mathfrak{g}))$ a highest weight vector. The action of $\gamma(U(\mathfrak{g}))$ by left multiplication on $\gamma(U(\mathfrak{g}))\mathsf{R} \cong V(\rho)$ defines a $\mathfrak{g}\times\mathfrak{g}$-equivariant algebra homomorphism $\gamma(U(\mathfrak{g})) \to \mathrm{End}(V(\rho))$. Since both sides are irreducible $\mathfrak{g} \times \mathfrak{g}$-representations, this map is an isomorphism. □

Proposition 3.4. *There is a unique isomorphism of $\mathfrak{g}\times\mathfrak{g}$-representations*

$$\mathrm{Cl}(\mathfrak{t}) \otimes \mathrm{End}(V(\rho)) \to \mathrm{Cl}(\mathfrak{g})$$

taking $x \otimes \mathrm{pr}_{V(\rho)^{\mathfrak{n}_+}}$ to $x\mathsf{R}$.

PROOF. Let $x \in \mathrm{Cl}(\mathfrak{t})$. Since $x\mathsf{R} = \mathsf{R}x$, we see that $\mathrm{Cl}(\mathfrak{t})\mathsf{R} \subset \mathrm{Cl}(\mathfrak{g})^{\mathfrak{n}_+\times\mathfrak{n}_-}$ is a space of highest weight vectors. As noted above, the highest weights are necessarily $(\rho, -\rho)$. Hence the $\mathfrak{g}\times\mathfrak{g}$-representation generated by this subspace has dimension

$$\dim(\mathrm{Cl}(\mathfrak{t}))\dim(\mathrm{End}(V(\rho))) = 2^{\dim\mathfrak{t}}2^{2|\mathfrak{R}_+|} = 2^{\dim\mathfrak{g}} = \dim\mathrm{Cl}(\mathfrak{g}).$$

This establishes that $\mathrm{Cl}(\mathfrak{t})\mathsf{R}$ is the entire space of highest weight vectors:

$$\mathrm{Cl}(\mathfrak{t})\mathsf{R} = \mathrm{Cl}(\mathfrak{g})^{\mathfrak{n}_+\times\mathfrak{n}_-},$$

and hence that $\mathrm{Cl}(\mathfrak{g}) \cong \mathrm{Cl}(\mathfrak{t}) \times \mathrm{End}(V(\rho))$ as $\mathfrak{g}\times\mathfrak{g}$-representations. □

Let us now restrict the $\mathfrak{g}\times\mathfrak{g}$-action on $\mathrm{Cl}(\mathfrak{g})$ to the diagonal $\mathfrak{g}$-action. By Schur's Lemma, the space $\mathrm{End}(V(\rho))^{\mathfrak{g}}$ of intertwing operators $V \to V$ is 1-dimensional, and is spanned by $\mathrm{Id}_{V(\rho)}$. Hence, by passing to $\mathfrak{g}$-invariants, Proposition 3.4 given an isomorphism of vector spaces, $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \cong \mathrm{Cl}(\mathfrak{t})$. In fact, one has:

THEOREM 3.5 (Kostant, Bazlov). *The Harish-Chandra projection p_{Cl} restricts to an isomorphism of algebras,*

$$p_{\mathrm{Cl}} \colon \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cl}(\mathfrak{t}). \tag{99}$$

PROOF. By §7, Proposition 5.9 the map p_{Cl} restricts to an algebra homomorphism on invariants. Hence, it suffices to show that (99) is a surjection onto $\mathrm{Cl}(\mathfrak{t})$.

Using Lemma 3.7 below, we can choose $a \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ with

$$a - 1 \in U^+(\mathfrak{n}_- \oplus \mathfrak{n}_+),$$

with the property

$$a.\,\mathrm{pr}_{V(\rho)^{\mathfrak{n}_+}} = \mathrm{Id}_{V(\rho)}.$$

The $\mathfrak{g}\times\mathfrak{g}$-equivariant isomorphism from Proposition 3.4 takes $\mathrm{Id}_{V(\rho)} \otimes x = a.(\mathrm{pr}_{V(\rho)^{\mathfrak{n}_+}} \otimes x)$ to

$$f(x) := a.(x\mathsf{R}) \in \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}.$$

Since

$$a - 1 \in U^+(\mathfrak{n}_- \oplus \mathfrak{n}_+),$$

and since p_{Cl} is an algebra morphism on $\mathfrak{t}$-invariants, it follows that

$$p_{\mathrm{Cl}}(f(x)) = p_{\mathrm{Cl}}(a.(x\mathsf{R})) = p_{\mathrm{Cl}}(x.\mathsf{R}) = p_{\mathrm{Cl}}(x)p_{\mathrm{Cl}}(\mathsf{R}) = x.$$

Here we used $p_{\mathrm{Cl}}(\mathsf{R}) = 1$. □

REMARK 3.6. Note that the proof gives an explicit inverse map, $f \colon \mathrm{Cl}(\mathfrak{t}) \to \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ to (99).

The proof of Theorem 3.5 used the following fact.

LEMMA 3.7. *Let* $\pi\colon \mathfrak{g} \to \mathrm{End}(V(\mu))$ *be an irreducible unitary representation of* $\mathfrak{g}$*, of highest weight* μ*. There exists* $a \in U(\mathfrak{g}) \times U(\mathfrak{g})$*, with* $a - 1 \in U^+(\mathfrak{n}_+ \oplus \mathfrak{n}_-)$*, such that the action of* a *on* $\mathrm{End}(V(\mu))$ *has the property*

$$a.\,\mathrm{pr}_{V(\mu)^{\mathfrak{n}_+}} = \mathrm{Id}_{V(\mu)}\,.$$

PROOF. Let v be a highest weight vector of $V(\mu)$, and $v^* \in V(\mu)^*$ the linear functional defined by Hermitian inner product with v. Then $\mathrm{pr}_{V(\mu)^{\mathfrak{n}_+}} = v\otimes v^*$. Let $v_1, \ldots, v_N$ be an orthonormal basis of $V(\mu)$, consisting of weight vectors, with $v_1 = v$ the highest weight vector. Let $a_1, \ldots, a_N \in U(\mathfrak{n}_-)$ be elements such that $v_i = \pi(a_i)v$, with $a_1 = 1$, and $a_i \in U^+(\mathfrak{n}_-)$ for $i > 1$. Let $v_i^* \in V(\rho)^*$ be the linear functionals defined by Hermitian inner product with v_i. Then $v_i^* = \pi^*(\overline{a}_i)v^*$, where $\overline{a}_i$ is the complex conjugate, and

$$\sum_{i=1}^N v_i \otimes v_i^* = \big(\sum_{i=1}^N \pi(a_i) \otimes \pi^*(\overline{a}_i)\big)(v \otimes v^*)$$

corresponds to the identity element $\mathrm{id}_{V(\rho)} \in \mathrm{End}(V(\rho)) \cong V(\rho) \otimes V(\rho)^*$. Hence

$$a := \sum_{i=1}^N a_i \otimes \overline{a}_i \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

has the desired property. □

We now obtain a sharper version of Proposition 3.4, as follows. Observe that $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ is the commutator of the subalgebra $\gamma(U(\mathfrak{g}))$.

THEOREM 3.8 (Kostant). *The multiplication map*

$$\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \otimes \gamma(U(\mathfrak{g})) \to \mathrm{Cl}(\mathfrak{g})$$

is an isomorphism of algebras.

PROOF. Since the two factors $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ and $\gamma(U(\mathfrak{g}))$ commute, and since $\dim(\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}) \dim(\gamma(U(\mathfrak{g}))) = \dim \mathrm{Cl}(\mathfrak{t}) \dim(\mathrm{End}(V(\rho))) = \dim \mathrm{Cl}(\mathfrak{g})$, it is enough to show that the product map is surjective.

For $y \in \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$, we have $y - p_{\mathrm{Cl}}(y) \in \mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$. By Theorem 3.5, it follows that

$$\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}\mathsf{R} = \mathrm{Cl}(\mathfrak{t})\mathsf{R}.$$

Using $\gamma(U(\mathfrak{g})) = \gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g}))$, this shows

$$\begin{aligned}\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}\gamma(U(\mathfrak{g})) &= \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}\gamma(U(\mathfrak{g}))\mathsf{R}\gamma(U(\mathfrak{g}))\\ &= \gamma(U(\mathfrak{g}))\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}\mathsf{R}\gamma(U(\mathfrak{g}))\\ &= \gamma(U(\mathfrak{g}))\mathrm{Cl}(\mathfrak{t})\mathsf{R}\gamma(U(\mathfrak{g}))\\ &= \mathrm{Cl}(\mathfrak{g}).\end{aligned}$$

□

REMARK 3.9. Theorem 3.8 was proved in Kostant's paper [**44**] by a different argument. As explained to us by Kostant in August 2003 at the Erwin Schr"odinger Institute, Theorem 3.8 implies Theorem 3.5. This was independently observed by Bazlov [**10**]. In our approach, we used a direct proof of Theorem 3.5 to obtain Theorem 3.8.

REMARK 3.10. Since the quantization map identifies $\mathrm{Cl}(\mathfrak{g}) \cong \wedge(\mathfrak{g})$ as $\mathfrak{g}$-representations, Proposition 3.4 also shows that

$$\wedge\mathfrak{g} \cong \wedge\mathfrak{t} \otimes \mathrm{End}(V(\rho))$$

as $\mathfrak{g}$-representations. Passing to invariants,

$$(\wedge\mathfrak{g})^{\mathfrak{g}} \cong \wedge\mathfrak{t}.$$

In particular, $\dim(\wedge\mathfrak{g})^{\mathfrak{g}} = 2^{\mathrm{rank}(\mathfrak{g})}$.

4. Equal rank subalgebras

We will consider a reductive Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ of equal rank. With no loss of generality, we may assume $\mathfrak{k}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$. Let $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$ be a Cartan subalgebra, given as the complexification of a maximal Abelian subalgebra $\mathfrak{t}_{\mathbb{R}} \subset \mathfrak{k}_{\mathbb{R}}$. The orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ will be denoted $\mathfrak{p}$.

EXAMPLES 4.1. For any $\xi \in \mathfrak{g}_{\mathbb{R}}$, the centralizer $\mathfrak{k} = \ker(\mathrm{ad}(\xi)) \subset \mathfrak{g}$ is an equal rank subalgebra. As extreme cases, one has $\mathfrak{k} = \mathfrak{t}$ and $\mathfrak{k} = \mathfrak{g}$. Other examples of equal rank subalgebras $\mathfrak{k} \subset \mathfrak{g}$ include:

(1) $\mathfrak{g}$ of type C_2 (i.e. $\mathfrak{sp}(4)$), $\mathfrak{k}$ of type $A_1 \times A_1$ (i.e. $\mathfrak{su}(2) \times \mathfrak{su}(2)$).
(2) $\mathfrak{g}$ of type G_2, $\mathfrak{k}$ of type A_2 (i.e. $\mathfrak{su}(3)$),
(3) $\mathfrak{g}$ of type F_4, $\mathfrak{k}$ of type B_4 (i.e. $\mathfrak{spin}(9)$),
(4) $\mathfrak{g}$ of type E_8, $\mathfrak{k}$ of type D_8 (i.e. $\mathfrak{spin}(16)$),

REMARK 4.2. A classification of semi-simple Lie subalgebras of a semi-simple Lie algebra was obtained by Dynkin [**27**], following earlier work of A. Borel and J. de Siebenthal [**11**] who classified equal rank subgroups of compact Lie groups. Dynkin's result may be summarized as follows. Let $\alpha_1, \ldots, \alpha_l$ be a set of simple roots for $\mathfrak{g}$, identified with the vertices of the Dynkin diagram. For each simple root α_i, there is an equal rank Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ having as its set of simple roots the α_j, $j \neq i$, together with the lowest root $\alpha_0 = -\alpha_{\max}$ of the simple summand $\mathfrak{g}'$ of $\mathfrak{g}$ containing the root space $\mathfrak{g}_{\alpha_i}$. That is, the Dynkin diagram of $\mathfrak{k}$ is obtained from that of $\mathfrak{g}$ by first replacing the component containing α_i (i.e. the Dynkin diagram of $\mathfrak{g}'$) with the *extended Dynkin diagram*, and then removing the vertex α_i to obtain an ordinary Dynkin diagram. [1] The resulting semi-simple equal rank subalgebra $\mathfrak{k}$ is a *maximal* Lie subalgebra of $\mathfrak{g}$ if and only if the coefficient

[1] Suppose $\mathfrak{g}$ is simple, and let $G_{\mathbb{R}}$ be the compact, simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$. Then the Lie algebras $\mathfrak{k}_{\mathbb{R}}$ obtained by this procedure are the Lie algebras of centralizers $K_{\mathbb{R}}$ of elements $g = \exp(\xi)$, where ξ is a vertex of the Weyl alcove of G. Up to conjugacy, these are precisely the centralizers that are semi-simple.

of α_i in the expression $\alpha_{\max} = \sum_j k_j\alpha_j$ is a prime number. Dynkin proved that any semi-simple equal rank $\mathfrak{k}$ is obtained by repeating this procedure a finite number of times. For details, see [**60**] or [**54**].

We denote by $\mathfrak{R}_{\mathfrak{k}} \subset \mathfrak{R}_{\mathfrak{g}} \subset \mathfrak{t}^*$ the set of roots of $\mathfrak{k} \subset \mathfrak{g}$, and let $\mathfrak{R}_{\mathfrak{p}}$ be its complement. The choice of a decomposition $\mathfrak{R}_{\mathfrak{g}} = \mathfrak{R}_{\mathfrak{g},+} \cup \mathfrak{R}_{\mathfrak{g},-}$ into positive and negative roots induces similar decompositions for $\mathfrak{R}_{\mathfrak{k}}$, $\mathfrak{R}_{\mathfrak{p}}$. We denote by

$$\rho_{\mathfrak{g}} = \tfrac{1}{2} \sum_{\alpha \in \mathfrak{R}_{\mathfrak{g},+}} \alpha$$

the half-sum of positive roots of $\mathfrak{g}$, and similarly define $\rho_{\mathfrak{k}}$ and $\rho_{\mathfrak{p}}$. Then

$$\rho_{\mathfrak{g}} = \rho_{\mathfrak{k}} + \rho_{\mathfrak{p}}.$$

Let $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ be the Lagrangian splitting of $\mathfrak{p}$ defined by the decomposition into positive and negative roots:

$$\mathfrak{p}_+ = \bigoplus_{\alpha \in \mathfrak{R}_{\mathfrak{p},+}} \mathfrak{g}_\alpha, \qquad \mathfrak{p}_- = \bigoplus_{\alpha \in \mathfrak{R}_{\mathfrak{p},-}} \mathfrak{g}_\alpha.$$

We will use it to define a spinor module

$$\mathsf{S}_{\mathfrak{p}} = \mathrm{Cl}(\mathfrak{p})/\mathrm{Cl}(\mathfrak{p})\mathfrak{p}_+ \cong \wedge\mathfrak{p}_-$$

over $\mathrm{Cl}(\mathfrak{p})$. Let $\varrho\colon \mathrm{Cl}(\mathfrak{p}) \to \mathrm{End}(\mathsf{S}_{\mathfrak{p}})$ be the Clifford action on the spinor module.

REMARK 4.3. The decomposition $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ (or equivalently the corresponding complex structure on $\mathfrak{p}_{\mathbb{R}}$) is not $\mathfrak{k}$-invariant, in general. Hence $\mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_-$ does not in general define a triangular decomposition in the sense of Section §7.5.1. It does define a triangular decomposition in case $\mathfrak{k}$ is the centralizer in $\mathfrak{g}$ of some element $\xi \in \mathfrak{t}_{\mathbb{R}}$.

The spinor module $\mathsf{S}_{\mathfrak{p}}$ carries a representation of $\mathfrak{k}$, defined by the Lie algebra homomorphism

$$\gamma_{\mathfrak{p}}\colon \mathfrak{k} \to \mathrm{Cl}(\mathfrak{p})$$

followed by teh Clifford action ϱ. We will refer to this action as the *spinor representation of* $\mathfrak{k}$ *on* $\mathsf{S}_{\mathfrak{p}}$.

On the other hand, $\wedge\mathfrak{p}_-$ carries a representation of $\mathfrak{t}$ by derivations of the super algebra structure, extending the adjoint action on $\mathfrak{p}_-$. We will denote this action by $\xi \mapsto \mathrm{ad}(\xi)$. The two representations are related as follows.

PROPOSITION 4.4. *Under the identification* $\mathsf{S}_{\mathfrak{p}} \cong \wedge\mathfrak{p}_-$*, the adjoint action of* $\mathfrak{t}$ *on* $\wedge\mathfrak{p}_-$ *and the spinor representation of* $\mathfrak{t} \subset \mathfrak{k}$ *on* $\mathsf{S}_{\mathfrak{p}}$ *are related by a* $\rho_{\mathfrak{p}}$*-shift:*

$$\varrho(\gamma_{\mathfrak{p}}(\xi)) = \mathrm{ad}(\xi) + \langle \rho_{\mathfrak{p}}, \xi \rangle, \quad \xi \in \mathfrak{t}.$$

PROOF. This is a special case of case (3) in §4.2.1, applied to $V = \mathfrak{p}_+$, with $A = \mathrm{ad}_\xi|_{\mathfrak{p}_+} \in \mathfrak{gl}(\mathfrak{p}_+) \subset \mathfrak{o}(\mathfrak{p})$. Here

$$\tfrac{1}{2}\,\mathrm{tr}(A) = \langle \rho_{\mathfrak{p}}, \xi \rangle. \qquad \square$$

Fix an ordering on $\mathfrak{R}_{\mathfrak{p},+}$. Then $\wedge\mathfrak{p}_-$ has a basis

$$v_X = \wedge_{\alpha\in X} e_{-\alpha}$$

where X ranges over subsets of $\mathfrak{R}_{\mathfrak{p},+}$.

PROPOSITION 4.5. *The weights for the $\mathfrak{k}$-action on $\mathsf{S}_\mathfrak{p}$ are the elements of the form*

$$\nu_X = \rho_\mathfrak{p} - \sum_{\alpha\in X}\alpha$$

where X ranges over subsets of $\mathfrak{R}_{\mathfrak{p},+}$. The multiplicity of a weight ν is equal to the number of subsets X such that $\nu = \nu_X$.

PROOF. The basis vector v_X is a weight vector for the adjoint action of $\mathfrak{t}$, with corresponding weight $-\sum_{\alpha\in X}\alpha$. It is thus also a weight vector for the Clifford action of $\mathfrak{t}$, with the weight shifted by $\rho_\mathfrak{p}$. $\square$

For any completely reducible $\mathfrak{t}$-representation on a finite-dimensional super space W, with weight spaces W_ν, define the formal character

$$\mathsf{ch}(W) = \mathsf{ch}(W^{\bar{0}}) - \mathsf{ch}(W^{\bar{1}}) = \sum_\nu (\dim(W_\nu^{\bar{0}}) - \dim(W_\nu^{\bar{1}}))e^\nu.$$

Basic properties of the formal character include,

$$\begin{aligned} \mathsf{ch}(W\oplus W') &= \mathsf{ch}(W) + \mathsf{ch}(W'),\\ \mathsf{ch}(W\otimes W') &= \mathsf{ch}(W)\mathsf{ch}(W'),\\ \mathsf{ch}(W^*) &= \mathsf{ch}(W)^*. \end{aligned}$$

The character for the adjoint representation on the super space $\wedge\mathfrak{p}_- = \bigotimes_{\alpha\in\mathfrak{R}_{\mathfrak{p},+}} \wedge(\mathbb{C}e_{-\alpha})$ is

$$\mathsf{ch}(\wedge\mathfrak{p}_-) = \prod_{\alpha\in\mathfrak{R}_{\mathfrak{p},+}} (1-\mathsf{e}^{-\alpha}),$$

and the character for the spinor representation on $\mathsf{S}_\mathfrak{p}$ is obtained from this by a $\rho_\mathfrak{p}$-shift. Thus

$$\begin{aligned} \mathsf{ch}(\mathsf{S}_\mathfrak{p}) &= \mathsf{e}^{\rho_\mathfrak{p}} \prod_{\alpha\in\mathfrak{R}_{\mathfrak{p},+}} (1-\mathsf{e}^{-\alpha})\\ &= \prod_{\alpha\in\mathfrak{R}_{\mathfrak{p},+}} (\mathsf{e}^{\alpha/2}-\mathsf{e}^{-\alpha/2}). \end{aligned}$$

REMARK 4.6. We can also view $\mathsf{S}_\mathfrak{p}$ as an *ungraded* $\mathfrak{g}$-representation. Repeating the calculation above without the signs, we find that its character is $\prod_{\alpha\in\mathfrak{R}_{\mathfrak{p},+}}(\mathsf{e}^{\alpha/2}+\mathsf{e}^{-\alpha/2})$. We claim that for $\mathfrak{k}=\mathfrak{t}$, this is in fact the character of the ρ-representation:

$$\mathsf{ch}(V(\rho)) = \prod_{\alpha\in\mathfrak{R}_+} (\mathsf{e}^{\alpha/2}+\mathsf{e}^{-\alpha/2}). \tag{100}$$

To see this, consider $\mathrm{Cl}(\mathfrak{g})\mathsf{R}$ as an *ungraded* $\mathfrak{g}$-representation, where $\mathfrak{g}$ acts by $\gamma\colon \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ followed by the left regular representation of $\mathfrak{g}$ on itself. As a representation of $\mathfrak{t} \subset \mathfrak{g}$, it breaks up into 2^l copies of the spinor representation of $\mathfrak{t}$ on $\mathsf{S}_{\mathfrak{t}^\perp}$:

$$\mathrm{Cl}(\mathfrak{g})\mathsf{R} = \mathrm{Cl}(\mathfrak{n}_-)\mathsf{R} \otimes \mathrm{Cl}(\mathfrak{t}) \cong \mathsf{S}_{\mathfrak{t}^\perp} \otimes \mathrm{Cl}(\mathfrak{t}),$$

hence the character is $2^l \prod_{\alpha\in\mathfrak{R}_+}(\mathsf{e}^{\alpha/2} + \mathsf{e}^{-\alpha/2})$. On the other hand, we had seen that $\mathrm{Cl}(\mathfrak{g})\mathsf{R}$ consists of 2^l copies of the ρ-representation, hence the character is $2^l\mathsf{ch}(V(\rho))$. Comparing, we obtain Equation (100). We see in particular that $\mathsf{S}_{\mathfrak{t}^\perp}$ is isomorphic to $V(\rho)$ as a $\mathfrak{t}$-representation.

Let $W_\mathfrak{k} \subset W_\mathfrak{g} = W$ be the Weyl groups of $\mathfrak{k} \subset \mathfrak{g}$. Define a set

$$W_\mathfrak{p} = \{w \in W \mid \mathfrak{R}_{\mathfrak{k},+} \subset w\mathfrak{R}_+\}$$

In terms of the Weyl chambers $\mathfrak{t}_+ \subset \mathfrak{t}_{\mathfrak{k},+}$ determined by the positive roots, the definition reads

$$W_\mathfrak{p} = \{w \in W \mid w\mathfrak{t}_+ \supset \mathfrak{t}_{\mathfrak{k},+}\}.$$

For any $w \in W$ denote by

$$\mathfrak{R}_{+,w} = \mathfrak{R}_+ \cap w\mathfrak{R}_-$$

the set of positive roots that become negative under w^{-1}.

Lemma 4.7. (1) *For all $w \in W$,*

$$\rho - w\rho = \sum_{\alpha\in\mathfrak{R}_{+,w}} \alpha.$$

If $X \subset \mathfrak{R}_+$ is a subset with $\rho - w\rho = \sum_{\alpha\in X}\alpha$, then $X = \mathfrak{R}_{+,w}$.

(2) *$w \in W_\mathfrak{p}$ if and only if $\mathfrak{R}_{+,w} \subset \mathfrak{R}_{\mathfrak{p},+}$.*

(3) *The map*

$$W_\mathfrak{k} \times W_\mathfrak{p} \to W,\quad (w_1, w_2) \to w_1 w_2$$

is a bijection; thus $W_\mathfrak{p}$ labels the left cosets of $W_\mathfrak{k}$ in W.

(4) *If $w \in W_\mathfrak{p}$, the element $w\rho - \rho_\mathfrak{k}$ is weight of $\mathsf{S}_\mathfrak{p}$ of multiplicity one.*

Proof. (1) The first part is Lemma 6.9 in Appendix B; the second part follows from Proposition 4.5, since $w\rho$ has multiplicity one. (See also Remark 4.6.)

(2) We have $\mathfrak{R}_{+,w} \subset \mathfrak{R}_{\mathfrak{p},+}$ if and only if the intersection

$$\mathfrak{R}_{+,w} \cap \mathfrak{R}_{\mathfrak{k},+} = w\mathfrak{R}_- \cap \mathfrak{R}_{\mathfrak{k},+}$$

is empty. But this means precisely that $\mathfrak{R}_{\mathfrak{k},+} \subset w\mathfrak{R}_+$, i.e. $w \in W_\mathfrak{p}$.

(3) Let $w \in W$ be given. Since $\mathfrak{R}_\mathfrak{k} \cap w\mathfrak{R}_+$ is a system of positive roots for $\mathfrak{k}$, there is a unique $w_1 \in W_\mathfrak{k}$ with

$$w_1\mathfrak{R}_{\mathfrak{k},+} = \mathfrak{R}_\mathfrak{k} \cap w\mathfrak{R}_+ \subset w\mathfrak{R}_+.$$

Thus $w_2 = w_1^{-1}w$ satisfies $\mathfrak{R}_{\mathfrak{k},+} \subset w_2\mathfrak{R}_+$, i.e. $w_2 \in W_\mathfrak{p}$.

(4) Suppose $w \in W_{\mathfrak{p}}$. By (1) and since $\rho = \rho_{\mathfrak{k}} + \rho_{\mathfrak{p}}$, we have
$$w\rho - \rho_{\mathfrak{k}} = \rho_{\mathfrak{p}} - \sum_{\alpha \in \mathfrak{R}_{+,w}} \alpha;$$
moreover $X = \mathfrak{R}_{+,w}$ is the unique subset for which this equation holds. Hence, by Proposition 4.5 $w\rho - \rho_{\mathfrak{k}}$ is a weight of multiplicity one.

□

We will now now assume that B is positive definite on $\mathfrak{g}_{\mathbb{R}}$, thus $||\mu||^2 = -B^*(\mu, \mu)$ for $\mu \in P \otimes_{\mathbb{Z}} \mathbb{R}$. We will find it convenient to introduce the notation $(\mu|\nu) = -B^*(\mu, \nu)$. A weight μ for a finite-dimensional $\mathfrak{g}$-representation V will be called a $\mathfrak{g}$-*extremal weight* if $||\mu + \rho||$ is maximal among all weights of V. Note that this notion depends on the choice of positive Weyl chamber (or equivalently, of $\mathfrak{R}_+$). The following Lemma is proved in Section 10 of Appendix B.

LEMMA 4.8. *If $\mu \in P(V)$ is a $\mathfrak{g}$-extremal weight, then the corresponding weight space is contained in the space of highest weight vectors: $V_\mu \subset V^{\mathfrak{n}_+}$. In particular, $V(\mu)$ appears in V with multiplicity equal to* $\dim V_\mu$.

In particular, for an irreducible $\mathfrak{g}$-representation the highest weight is the unique $\mathfrak{g}$-extremal weight. As another example, the weights $w\rho - \rho_{\mathfrak{k}}$, $w \in W_{\mathfrak{p}}$ for the $\mathfrak{k}$-representation on $S_{\mathfrak{p}}$ are exactly the $\mathfrak{k}$-extremal of $S_{\mathfrak{p}}$. Indeed, if ν is any weight of $S_{\mathfrak{p}}$, then $\nu + \rho_{\mathfrak{k}}$ is a weight of $V(\rho)$. Hence $||nu + \rho_{\mathfrak{k}}|| \le ||\rho||$. Equality holds if $\nu + \rho_{\mathfrak{k}} = w\rho$ for some $w \in W$, but as we saw $w\rho - \rho_{\mathfrak{k}}$ is a weight of $S_{\mathfrak{p}}$ if and only if $w \in W_{\mathfrak{p}}$. More generally we have:

PROPOSITION 4.9. *Let $\mathfrak{k} \subset \mathfrak{g}$ as above. For any $\mathfrak{g}$-dominant weight $\mu \in P_+$,*
$$w(\mu + \rho) - \rho_{\mathfrak{k}}, \quad w \in W_{\mathfrak{p}}$$
are $\mathfrak{k}$-extremal weights for the $\mathfrak{k}$-representation $V(\mu) \otimes \mathsf{S}_{\mathfrak{p}}$, each appearing with multiplicity one. The irreducible $\mathfrak{k}$-representation with highest weight $w(\mu + \rho) - \rho_{\mathfrak{k}}$ appears in the even (resp. odd) component, depending on the parity of $l(w)$.

PROOF. Write $V = V(\mu)$. The weights of $V \otimes \mathsf{S}_{\mathfrak{p}}$ are sums $\nu = \nu_1 + \nu_2$, where ν_1 is a weight of V and ν_2 is a weight of $\mathsf{S}_{\mathfrak{p}}$. Given such a weight, choose $w \in W$ such that $w^{-1}(\nu + \rho_{\mathfrak{k}})$ lies in the positive chamber for $\mathfrak{g}$. That is,
$$(w^{-1}(\nu + \rho_{\mathfrak{k}})|\alpha) \ge 0 \tag{101}$$
for all $\alpha \in \mathfrak{R}_+$. Since $\rho_{\mathfrak{k}}$ is a weight of $\mathsf{S}_{\mathfrak{k} \cap \mathfrak{t}^\perp}$, the sum $\nu_2 + \rho_{\mathfrak{k}}$ is among the $\mathfrak{t}$-weights of $\mathsf{S}_{\mathfrak{t}^\perp} = \mathsf{S}_{\mathfrak{p}} \otimes \mathsf{S}_{\mathfrak{k} \cap \mathfrak{t}^\perp}$, i.e. it lies in $P(V(\rho))$. Hence also $w^{-1}(\nu_2 + \rho_{\mathfrak{k}}) \in P(V(\rho))$. It follows that
$$\rho = w^{-1}(\nu_2 + \rho_{\mathfrak{k}}) + \sum_{\alpha \in \mathfrak{R}_+} k_\alpha \alpha$$

with $k_\alpha \geq 0$. Similarly, since $w^{-1}\nu_1$ is a weight of V,

$$\mu = w^{-1}\nu_1 + \sum_{\alpha \in \mathfrak{R}_+} l_\alpha \alpha$$

where $l_\alpha \geq 0$. Adding, we obtain

$$\mu + \rho = w^{-1}(\nu + \rho_{\mathfrak{k}}) + \sum_{\alpha \in \mathfrak{R}_+} (k_\alpha + l_\alpha)\alpha.$$

Using Equation (101), we obtain

$$||\mu + \rho|| \geq ||w^{-1}(\nu + \rho_{\mathfrak{k}})|| = ||\nu + \rho_{\mathfrak{k}}||,$$

with equality if and only if all l_α, k_α are zero. The latter case is equivalent to $\nu_1 = w\mu$ and $\nu_2 = w\rho - \rho_{\mathfrak{k}}$. Since these are indeed weights for V resp. $\mathsf{S}_{\mathfrak{p}}$, it follows that their sum $\nu := w(\mu+\rho) - \rho_{\mathfrak{k}}$ is a $\mathfrak{k}$-extremal weight for $V \otimes \mathsf{S}_{\mathfrak{p}}$. Since $\nu + \rho_{\mathfrak{k}}$ lies in the interior of the positive chamber for $\mathfrak{k}$, while $\mu + \rho$ lies in the interior of the positive chamber for $\mathfrak{g}$, the equality $w(\mu + \rho) = \nu + \rho_{\mathfrak{k}}$ shows $w \in W_{\mathfrak{p}}$. Note also that w is uniquely determined by the equation $w(\mu+\rho) = \nu+\rho_{\mathfrak{k}}$. Suppose conversely that $w \in W_{\mathfrak{p}}$ is given, and that ν_1, ν_2 are weights for V, $\mathsf{S}_{\mathfrak{p}}$ with $\nu_1+\nu_2 = w(\mu+\rho)-\rho_{\mathfrak{k}}$. The argument above shows that $\nu_1 = w'\mu$, $\nu_2 = w'\rho - \rho$ for some $w' \in W$. But then $w(\mu+\rho) = w'(\mu+\rho)$, thus $w' = w$. This shows that ν_1, ν_2 are uniquely determined. It follows that the weight $w(\mu+\rho)-\rho_{\mathfrak{k}}$ of $V \otimes \mathsf{S}_{\mathfrak{p}}$ has multiplicity one, and the corresponding weight space is just the tensor product $V_{w\mu} \otimes (\mathsf{S}_{\mathfrak{p}})_{w\rho-\rho_{\mathfrak{k}}}$. The weight space has even (resp. odd) parity if and only if $(\mathsf{S}_{\mathfrak{p}})_{w\rho-\rho_{\mathfrak{k}}}$ has even (resp. odd) parity, if and only if $l(w)$ is even (resp. odd). ☐

5. The kernel of D_V

We keep our assumption that B is the complexification of a *positive definite* invariant symmetric bilinear form on $\mathfrak{g}_{\mathbb{R}}$. Then $\mathsf{S}_{\mathfrak{p}}$ acquires a Hermitian structure, and $\mathrm{Cl}_{\mathfrak{p}}$ acts unitarily. Fix an irreducible unitary $\mathfrak{g}$-representation $V = V(\mu)$ of highest weight μ. Then $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$ acts on $V \otimes \mathsf{S}_{\mathfrak{p}}$, and hence the relative Dirac operator $\mathcal{D}(\mathfrak{g}, \mathfrak{k}) \in (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^{\mathfrak{k}}$ is represented as a $\mathfrak{k}$-equivariant, skew-adjoint odd operator

$$D_V \in \mathrm{End}(V \otimes \mathsf{S}_{\mathfrak{p}}).$$

We are interested in the kernel of D_V. Denote by $M(\nu)$ the irreducible $\mathfrak{k}$-representation labeled by a dominant $\mathfrak{k}$-weight ν.

THEOREM 5.1 (Kostant [**45**]). *As a $\mathfrak{k}$-representation, the kernel of the cubic Dirac operator on $V \otimes \mathsf{S}_{\mathfrak{p}}$ is a direct sum*

$$\ker(D_V) \cong \bigoplus_{w \in W_{\mathfrak{p}}} M(w(\mu + \rho) - \rho_{\mathfrak{k}}).$$

Then even (resp. odd) part are given as sums over $w \in W_{\mathfrak{p}}$ with $l(w)$ even (resp. odd).

PROOF. Since D_V is skew-adjoint, its kernel coincides with the kernel of its square. We will hence determine $\ker(D_V^2)$. Consider the decomposition

$$V \otimes \mathsf{S}_{\mathfrak{p}} = \bigoplus_{\nu} (V \otimes \mathsf{S}_{\mathfrak{p}})_{[\nu]}$$

into the $\mathfrak{k}$-isotypical components, labeled by $\mathfrak{k}$-dominant weights. (We use a subscript $[\nu]$ to avoid confusion with the weight space.) Since D_V is $\mathfrak{k}$-invariant, it preserves each of these components. Using the formula

$$\mathcal{D}(\mathfrak{g},\mathfrak{k})^2 = \mathrm{Cas}_{\mathfrak{g}} - j(\mathrm{Cas}_{\mathfrak{k}}) - ||\rho||^2 + ||\rho_{\mathfrak{k}}||^2$$

(cf. Equation (88) and §8, Proposition 2.5), together with the formula for the action of the Casimir element in an irreducible representation (cf. §8, Proposition 2.1), it follows that the operator D_V^2 acts on each $\mathfrak{k}$-isotypical subspace $(V \otimes \mathsf{S}_{\mathfrak{p}})_{[\nu]}$ as a scalar,

$$-||\mu+\rho||^2 + ||\nu+\rho_{\mathfrak{k}}||^2.$$

In particular, $\ker(D_V^2)$ is the sum over all $\mathfrak{k}$-isotypical components for which $||\mu+\rho|| = ||\nu+\rho_{\mathfrak{k}}||$. Proposition 4.9 shows that the corresponding weights ν appear with multiplicity one, and exactly the weights $\nu = w(\mu+\rho)-\rho_{\mathfrak{k}}$ with $w \in W_{\mathfrak{p}}$. (We see once again that these weights are $\mathfrak{k}$-extremal, since D_V is skew-adjoint and hence D_V^2 is non-positive.) $\square$

Following Gross-Kostant–Ramond–Sternberg [**31**] and Kostant [**45**], we will refer to the collection of irreducible representations

$$M(w(\mu+\rho)-\rho_{\mathfrak{k}}), \quad w \in W_{\mathfrak{p}}$$

as the *multiplet* indexed by μ. Note that an irreducible $\mathfrak{k}$-representation of highest weight ν belongs to some multiplet if and only if $\nu+\rho_{\mathfrak{k}}$ is a $\mathfrak{g}$-weight which furthermore is regular for the W-action.

THEOREM 5.2 (Kostant [**45**]). *The dimensions of the irreducible representations in each multiplet satisfy*

$$\sum_{w \in W_{\mathfrak{p}}} (-1)^{l(w)} \dim M(w(\mu+\rho)-\rho_{\mathfrak{k}}) = 0.$$

PROOF. The exact sequence

$$0 \to \ker(D_V)^{\bar{0}} \to (V \otimes \mathsf{S}_{\mathfrak{p}})^{\bar{0}} \xrightarrow{D_V} (V \otimes \mathsf{S}_{\mathfrak{p}})^{\bar{1}} \to \mathrm{coker}(D_V)^{\bar{1}} \to 0 \tag{102}$$

together with $\mathrm{coker}(D_V)^{\bar{1}} \cong \ker(D_V)^{\bar{1}}$ gives $\dim(\ker(D_V)^{\bar{0}}) = \dim(\ker(D_V)^{\bar{1}})$. $\square$

Using §8, Proposition 2.1, it is immediate that $\mathrm{Cas}_{\mathfrak{k}}$ acts on the multiplet indexed by μ as a constant $-||\mu+\rho||^2 + ||\rho_{\mathfrak{k}}||^2$. In fact, much more is true. Let $U(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{k})^{\mathfrak{k}}$ correspond to the inclusion $(S\mathfrak{g})^{\mathfrak{g}} \to (S\mathfrak{k})^{\mathfrak{k}}$ under the Duflo isomorphism. (Cf. §7, Theorem 4.12.)

THEOREM 5.3. [**31, 45**] *Let $y \in (U\mathfrak{k})^{\mathfrak{k}}$ be an element in the image of $U(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{k})^{\mathfrak{k}}$. Then the action of y on the members of any multiplet*

$$M(w(\mu+\rho)-\rho_{\mathfrak{k}}), \quad w \in W_{\mathfrak{p}}$$

is independent of $w \in W_{\mathfrak{p}}$.

PROOF. Let $x \in U(\mathfrak{g})^{\mathfrak{g}}$ be given, and $y \in U(\mathfrak{k})^{\mathfrak{k}}$ its image. By §7, Theorem 4.12, there exists $z \in \mathcal{W}(\mathfrak{g},\mathfrak{k}) = (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^{\mathfrak{k}}$ with

$$x - j(y) = [\mathcal{D}(\mathfrak{g},\mathfrak{k}), z],$$

where $j\colon \mathcal{W}\mathfrak{k} \to \mathcal{W}\mathfrak{g}$ is a morphism of $\mathfrak{k}$-differential algebra defined in that section. Under the action of $\mathcal{W}(\mathfrak{g},\mathfrak{k})$ on $\phi \in \ker(D_V) \subset V \otimes \mathsf{S}_{\mathfrak{p}}$, this identity gives

$$x.\phi - j(y).\phi = D_V(z.\phi).$$

Since $\mathcal{D}(\mathfrak{g},\mathfrak{k}) \in \mathcal{W}(\mathfrak{g},\mathfrak{k})$ commutes with elements of $j(\mathcal{W}\mathfrak{k})$, the action of $j(y)$ preserves $\ker(D_V)$. On the other hand, $\mathcal{D}(\mathfrak{g},\mathfrak{k})$ also commutes with elements of $(U\mathfrak{g})^{\mathfrak{g}} \subset \mathcal{W}\mathfrak{g}$, hence also x preserves $\ker(D_V)$. We conclude that $x.\phi - j(y).\phi \in \ker(D_V)$, hence $z.\phi \in \ker(D_V^2) = \ker(D_V)$. This proves $j(y).\phi = x.\phi$. But the action of $x \in (U\mathfrak{g})^{\mathfrak{g}}$ on $\ker(D_V)$ is the scalar by which x acts on $V(\mu)$. $\square$

REMARK 5.4. As announced by Gross-Kostant-Ramond-Sternberg [**31**] and proved in Kostant's article [**45**], this property characterizes the triplets: Among the $\mathfrak{k}$-dominant weights ν such that $\nu+\rho_{\mathfrak{k}}$ is a regular weight for $\mathfrak{g}$, any triplet is determined by the values of elements in the image of $U(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{k})^{\mathfrak{k}}$.

Another appplication is the following generalized Weyl character formula.

THEOREM 5.5 (Gross-Kostant-Ramond-Sternberg [**31**]). *Let $V(\mu)$ be the irreducible $\mathfrak{g}$-representation of highest weight μ. Then*

$$\mathsf{ch}(V(\mu)) = \frac{\sum_{w \in W_{\mathfrak{p}}} (-1)^{l(w)} \mathsf{ch}(M(w(\mu+\rho)-\rho_{\mathfrak{k}}))}{\prod_{\alpha \in \mathfrak{R}_{\mathfrak{p},+}} (\mathsf{e}^{\alpha/2} - \mathsf{e}^{-\alpha/2})}.$$

PROOF. Write $V = V(\mu)$. The exact sequence (102) shows that

$$\mathsf{ch}(V \otimes \mathsf{S}_{\mathfrak{p}}) = \mathsf{ch}(\ker(D_V)).$$

But

$$\mathsf{ch}(V \otimes \mathsf{S}_{\mathfrak{p}}) = \mathsf{ch}(V)\mathsf{ch}(\mathsf{S}_{\mathfrak{p}}) = \mathsf{ch}(V) \prod_{\alpha \in \mathfrak{R}_{\mathfrak{p},+}} (\mathsf{e}^{\alpha/2} - \mathsf{e}^{-\alpha/2}),$$

$$\mathsf{ch}(\ker(D_V)) = \sum_{w \in W_{\mathfrak{p}}} (-1)^{l(w)} \mathsf{e}^{w(\mu+\rho)-\rho_{\mathfrak{k}}}. \qquad \square$$

If $\mu = 0$, one obviously has $\mathsf{ch}(V(0)) = 1$. Hence

COROLLARY 5.6.

$$\prod_{\alpha\in\mathfrak{R}_{\mathfrak{p},+}} (\mathsf{e}^{\alpha/2}-\mathsf{e}^{-\alpha/2}) = \sum_{w\in W_{\mathfrak{p}}} (-1)^{l(w)}\mathsf{e}^{w\rho-\rho_{\mathfrak{k}}}$$

For $\mathfrak{k}=\mathfrak{t}$, the GKRS character formula specializes to the usual Weyl character formula:

$$\mathsf{ch}(V(\mu)) = \frac{\sum_{w\in W}(-1)^{l(w)}\mathsf{e}^{w(\mu+\rho)}}{\prod_{\alpha\in\mathfrak{R}_+}(\mathsf{e}^{\alpha/2}-\mathsf{e}^{-\alpha/2})}. \tag{103}$$

6. q-dimensions

Kac, Möseneder-Frajria and Papi [**39**, Proposition 5.9] discovered that the dimension formula for multiplets generalizes to 'q-dimensions'. We recall the definition: Let $\mathfrak{g}$ be a semi-simple Lie algebra with given choice of $\mathfrak{R}_+$, and $\alpha_1,\ldots,\alpha_l$ the corresponding simple roots. Let $\rho^\vee$ be the half-sum of positive co-roots $\alpha^\vee$. Alternatively, $\rho^\vee$ is characterized by its property

$$\langle\alpha,\rho^\vee\rangle = 1$$

for every simple root α of $\mathfrak{g}$. The q-dimension of a $\mathfrak{g}$-representation is defined as the polynomial in q,

$$\dim_q(V) = \sum_\nu \dim(V_\mu) q^{\langle\mu,\rho^\vee\rangle}$$

(Other normalizations exist in the literature.) One has the following formula:

PROPOSITION 6.1. *The q-dimension of the irreducible representation $V(\mu)$ of highest weight $\mu\in P_+$ is given by the formula,*

$$\dim_q(V(\mu)) = \prod_{\alpha\in\mathfrak{R}_+} \frac{[\langle\mu+\rho,\alpha^\vee\rangle]_q}{[\langle\rho,\alpha^\vee\rangle]_q}$$

with the q-integers

$$[n]_q = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}.$$

PROOF. By the Weyl character formula (103),

$$\begin{aligned}\dim_q(V(\mu)) &= \frac{\sum_{w\in W}(-1)^{l(w)}q^{\langle w(\mu+\rho),\rho^\vee\rangle}}{\sum_{w\in W}(-1)^{l(w)}q^{\langle w\rho,\rho^\vee\rangle}}\\ &= \frac{\sum_{w\in W}(-1)^{l(w)}q^{\langle\mu+\rho,w^{-1}\rho^\vee\rangle}}{\sum_{w\in W}(-1)^{l(w)}q^{\langle\rho,w^{-1}\rho^\vee\rangle}}\\ &= \prod_{\alpha\in\mathfrak{R}_+}\frac{q^{\langle\mu+\rho,\alpha^\vee\rangle/2}-q^{-\langle\mu+\rho,\alpha^\vee\rangle/2}}{q^{\langle\rho,\alpha^\vee\rangle/2}-q^{-\langle\rho,\alpha^\vee\rangle/2}}\\ &= \prod_{\alpha\in\mathfrak{R}_+}\frac{[\langle\mu+\rho,\alpha^\vee\rangle]_q}{[\langle\rho,\alpha^\vee\rangle]_q}.\end{aligned}$$

□

For $q \to 1$, one has $\lim_{q\to 1}[n]_q = n$, and one recovers the *Weyl dimension formula*,

$$\dim(V(\mu)) = \prod_{\alpha\in\mathfrak{R}_+} \frac{\langle \mu+\rho, \alpha^\vee\rangle}{\langle \rho, \alpha^\vee\rangle}.$$

(Indeed, the proof above is the standard proof of the dimension formula, cf. Duistermaat-Kolk [**26**, Chapter 4.9].) As a special case, note that

$$\dim(V(\rho)) = 2^{|\mathfrak{R}_+|}.$$

LEMMA 6.2. [**39**, Lemma 5.8] *Suppose $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as above, where $\mathfrak{g}$ is a semi-simple Lie algebra, and $\mathfrak{k} \subset \mathfrak{g}$ is a semi-simple subalgebra of equal rank. There exists a root $\alpha \in \mathfrak{R}_\mathfrak{p}$ with*

$$\langle \alpha, \rho_\mathfrak{k}^\vee\rangle = 0.$$

More generally, if $\mathfrak{k} \subset \mathfrak{g}$ are equal rank reductive Lie algebras, there exists an element $\tilde\rho_\mathfrak{k} \in \mathfrak{t}$ (not unique) such that $\langle \alpha, \tilde\rho_\mathfrak{k}\rangle = 1$ for all simple roots of $\mathfrak{k}$, and $\langle \alpha, \tilde\rho_\mathfrak{k}\rangle = 0$ for at least one $\alpha \in \mathfrak{R}_\mathfrak{p}$.

PROOF. We first assume $\mathfrak{k}$ semi-simple. Consider a chain $\mathfrak{k} = \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ of subalgebras with $\mathfrak{g}_i$ maximal in $\mathfrak{g}_{i+1}$. Then each $\mathfrak{g}_i$ is a semi-simple subalgebra, and the set of roots of $\mathfrak{g}_i$ is a subset of those of $\mathfrak{g}$. We may thus assume that $\mathfrak{k}$ is maximal in $\mathfrak{g}$. Furthermore, by splitting $\mathfrak{g}$ into its simple components we may assume that $\mathfrak{g}$ is simple. Let $\alpha_1, \ldots, \alpha_l$ be the simple roots, and let $\alpha_{\max} = \sum_j k_j\alpha_j$ the highest root. The coefficients k_j are called the Dynkin marks. According to results of Borel-de Siebenthal [**11**], Dynkin [**27**] and Tits [**60**], the maximal[2] equal rank semi-simple subalgebras are classified (up to conjugacy) by the set of all $i \in \{1, \ldots, l\}$ for which the Dynkin mark k_i is prime. More precisely, the set of simple roots of the subalgebra is

$$\{\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_l\}$$

with $\alpha_0 = -\alpha_{\max}$. Writing $\alpha_i = \frac{1}{k_i}(\alpha_{\max} - \sum_{j\neq i} k_j\alpha_j)$ we obtain

$$\langle \alpha_i, \rho_\mathfrak{k}^\vee\rangle = \frac{1}{k_i}(-1 - \sum_{j\neq i} k_j) = 1 - \frac{h}{k_i}$$

where $h = 1 + \langle \alpha_{\max}, \rho^\vee\rangle = 1 + \sum_j k_j$ is the Coxeter number. It is a standard fact (which may be verified e.g. by consulting the tables for simple Lie algebras) that every prime Dynkin labels k_i divides the Coxeter number h. Hence $\langle \alpha_i, \rho_\mathfrak{k}^\vee\rangle \in \mathbb{Z}$. This proves the existence of a root $\alpha \in \mathfrak{R}_\mathfrak{p}$ with $\langle \alpha, \rho_\mathfrak{k}^\vee\rangle \in \mathbb{Z}_{\geq 0}$. If $\langle \alpha, \rho_\mathfrak{k}^\vee\rangle > 0$, then $\langle \alpha, \beta^\vee\rangle > 0$ for some simple root β of $\mathfrak{k}$. Thus $\alpha - \beta$ is a root. Since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, one has $\alpha - \beta \in \mathfrak{R}_\mathfrak{p}$, and

$$\langle \alpha - \beta, \rho_\mathfrak{k}^\vee\rangle = \langle \alpha, \rho_\mathfrak{k}\rangle - 1.$$

[2] A Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is called *maximal* if it is not contained in a Lie subalgebra other than $\mathfrak{k}, \mathfrak{g}$.

By repeating this procedure, one eventually finds the desired root in $\mathfrak{R}_{\mathfrak{p}}$ whose pairing with $\rho_{\mathfrak{k}}^\vee$ is zero. This proves the Lemma for $\mathfrak{g}, \mathfrak{k}$ semi-simple. If $\mathfrak{k}$ is not semi-simple, let $\mathfrak{z} \neq 0$ be its center. Choose any root $\alpha \in \mathfrak{R}_{\mathfrak{p}}$ that is not orthogonal to $\mathfrak{z}$. For suitable $\xi \in \mathfrak{z}$, we then have

$$\langle \alpha, \xi \rangle = \langle \alpha, \rho_{\mathfrak{k}}^\vee \rangle,$$

so that $\widetilde{\rho}_{\mathfrak{k}}^\vee = \rho_{\mathfrak{k}}^\vee - \xi$ has the desired properties. $\square$

THEOREM 6.3 (Kac, Möseneder-Frajria and Papi [**39**]). *Suppose $\mathfrak{k}$ is an equal rank semi-simple subalgebra of $\mathfrak{g}$. The q-dimensions of the irreducible $\mathfrak{k}$-representations in each multiplet satisfy*

$$\sum_{w \in W_{\mathfrak{p}}} (-1)^{l(w)} \dim_q M(w(\mu + \rho) - \rho_{\mathfrak{k}}) = 0.$$

PROOF. By the generalized Weyl character formula, Theorem 5.5, the left hand side of the displayed equation equals

$$\dim_q(V) \prod_{\alpha \in \mathfrak{R}_{\mathfrak{p},+}} (q^{\langle \alpha, \rho_{\mathfrak{k}}^\vee \rangle /2} - q^{\langle \alpha, \rho_{\mathfrak{k}}^\vee \rangle /2})$$

where V is viewed as a $\mathfrak{k}$-representation by restriction. By Lemma 6.2, at least one factor in the product over $\mathfrak{R}_{\mathfrak{p},+}$ is zero. $\square$

As pointed out in [**39**], the results extends to arbitrary equal rank reductive subalgebras $\mathfrak{k} \subset \mathfrak{g}$, provided the q-dimension is defined using $\widetilde{\rho}_{\mathfrak{k}}^\vee$ (cf. Lemma 6.2) instead of $\rho_{\mathfrak{k}}^\vee$.

7. The shifted Dirac operator

Return to the full Dirac operator $\mathcal{D} \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$ for a reductive Lie algebra $\mathfrak{g}$. (We still assume that B is positive definite on $\mathfrak{g}_{\mathbb{R}}$.) Fix a unitary module S over $\mathrm{Cl}(\mathfrak{g})$, and let $V = V(\mu)$ be an irreducible unitary $\mathfrak{g}$-representation of highest weight μ. Then $\mathcal{D}$ becomes a skew-adjoint odd operator D_V on $V \otimes \mathsf{S}$. Since $\mathcal{D}^2 = \mathrm{Cas}_{\mathfrak{g}} - ||\rho||^2$, the action on $V \otimes \mathsf{S}$ is as a scalar, $-||\mu + \rho||^2$. In particular, D is invertible as an operator on $V \otimes \mathsf{S}$.

As noted by Freed-Hopkins-Teleman [**29**], one obtains interesting results by shifting the Dirac operator by elements $\tau \in \sqrt{-1}\mathfrak{g}$:

$$\mathcal{D}_\tau = \mathcal{D} - \tau \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$$

Note that $\mathcal{D}_\tau$ no longer squares to a central element, but instead satisfies

$$\mathcal{D}_\tau^2 = \mathrm{Cas} - ||\rho||^2 + B(\tau, \tau) - 2(\widehat{\tau} + \gamma(\tau)).$$

We denote by $\tau^* = B^\flat(\tau) \in \sqrt{-1}\mathfrak{g}^*$ the image of τ under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ defined by B. Then $B(\tau, \tau) = B^*(\tau^*, \tau^*) = -||\tau^*||^2$. The element $\mathcal{D}_\tau$ is represented on $V \otimes \mathsf{S}$, as a skew-adjoint operator. The identity above becomes

$$\mathcal{D}_\tau^2 \Big|_{V \otimes \mathsf{S}} = -||\mu + \rho||^2 - ||\tau^*||^2 - 2L(\tau).$$

where $L(\tau)$ indicates the diagonal action of $\tau \in \mathfrak{g}$ on $V \otimes \mathsf{S}$.

Proposition 7.1 (Freed-Hopkins-Teleman [**29**]). *Let $V = V(\mu)$ be the irreducible representation of highest weight $\mu \in P_+$. For $\tau \in \sqrt{-1}\mathfrak{g}$, the operator $\mathcal{D}_\tau$ on $V \otimes \mathsf{S}$ is invertible, unless τ^* lies in the coadjoint orbit of $\mu + \rho$. Moreover, for $\tau^* = \mu + \rho$ the kernel of $\mathcal{D}_\tau$ is on $V \otimes \mathsf{S}$ is the weight space $(V \otimes \mathsf{S})_{\mu+\rho} = V_\mu \otimes \mathsf{S}_\rho$.*

Proof. Since the map $\tau \mapsto \mathcal{D}_\tau$ is $G_\mathbb{R}$-equivariant, we may assume that τ lies in the positive Weyl chamber. In particular, $\mathcal{D}_\tau$ is then $\mathfrak{t}$-equivariant, and hence preserves all weight spaces. On the weight space $(V \otimes \mathsf{S})_\nu \neq 0$ the operator $\mathcal{D}_\tau^2$ acts as

$$\mathcal{D}_\tau^2\big|_{(V\otimes\mathsf{S})_\nu} = -||\mu+\rho||^2 - ||\tau^*||^2 - 2\langle\nu,\tau\rangle = -||\mu+\rho-\tau^*||^2 + 2\langle\mu+\rho-\nu,\tau\rangle.$$

The weight ν is a sum of weights ν_1 of V and ν_2 of S. Both $\mu-\nu_1$ and $\rho-\nu_2$ are linear combinations of positive roots with non-negative coefficients, hence so is $\mu+\rho-\nu$. Hence both terms in the formula for $\mathcal{D}_\tau^2\big|_{(V\otimes\mathsf{S})_\nu}$ are ≤ 0. Hence D_τ^2 is non-zero on the weight space $(V \otimes \mathsf{S})_\nu$ unless

$$\mu + \rho - \tau^* = 0, \quad \langle\mu + \rho - \nu, \tau\rangle = 0.$$

But $\tau^* = \mu + \rho$ implies that τ lies in the interior of the Weyl chamber, and hence the condition $\langle\mu + \rho - \nu, \tau\rangle = 0$ forces $\nu = \mu + \rho$. It follows that the kernel of D_τ^2 on $V \otimes \mathsf{S}$ is the highest weight space, $(V \otimes \mathsf{S})_{\mu+\rho} = V_\mu \otimes \mathsf{S}_\rho$. $\square$

The Proposition shows that the family of Dirac operators $\tau^* \mapsto \mathcal{D}_\tau$ on $\sqrt{-1}\mathfrak{g}_\mathbb{R}^*$ (viewed a a representative for an equivariant K-theory class) has support the coadjoint orbit of $G_\mathbb{R}.(\mu + \rho)$. As shown in [**29**], the $G_\mathbb{R}$-equivariant K-theory class of this family of operators is identified with the class in $K_G(\mathrm{pt}) = R(G)$ defined by the representation $[V]$.

8. Dirac induction

In contrast to the previous sections, we will denote by $\mathfrak{t}, \mathfrak{k}, \mathfrak{g}, \ldots$ compact real Lie algebras, and $\mathfrak{t}^\mathbb{C}, \mathfrak{k}^\mathbb{C}, \mathfrak{g}^\mathbb{C}, \ldots$ the reductive Lie algebras obtained by complexification. The corresponding Lie groups will be denoted $T, K, G, \ldots$ and $T^\mathbb{C}, K^\mathbb{C}, G^\mathbb{C}, \ldots$ respectively. We will discuss 'Dirac induction' from twisted representations of equal rank subgroups $K \subset G$ of a compact Lie group, using the cubic Dirac operator. It may be viewed as analogous to the process of *holomorphic induction*, but works in more general settings since G/K need not carry an invariant complex structure. This Section draws from the papers by Kostant [**45, 46**] as well as from [**61, 50, 38**].

8.1. Central extensions of compact Lie groups. We will need some background material on central extensions of Lie groups G. In this Section, *central extension* will always mean a central extension by the circle group,

$$1 \to \mathrm{U}(1) \to \widehat{G} \to G \to 1.$$

We will refer to the image of $\mathrm{U}(1)$ in this sequence as *the central* $\mathrm{U}(1)$ *(in* $\widehat{G}$*)*, even though there may be other $\mathrm{U}(1)$ subgroups of the center.

A *morphism of central extensions* is given by a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{U}(1) \arrow[r] \arrow[d, "="] & \widehat{K} \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathrm{U}(1) \arrow[r] & \widehat{G} \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

where the vertical maps are group homomorphisms. If a morphism $K \to G$ is given, one can use the diagram to define a *pull-back* of a central extension of G to a central extension of K.

An *automorphism* of a central extension $\widehat{G}$ is a morphism from the central extension to itself. That is, it is a Lie group automorphism of $\widehat{G}$ commuting with the action of the central U(1). The automorphism is called a *gauge transformation* of $\widehat{G}$ if the underlying group homomorphism of G is the identity. It is easy to see that the group of gauge transformations of a central extension $\widehat{G}$ is the abelian group

$$\mathrm{Hom}(G, \mathrm{U}(1))$$

of all group homomorphisms $G \to \mathrm{U}(1)$. More generally, any two morphisms of central extensions $\widehat{K} \to \widehat{G}$ with a given underlying map $K \to G$ are related by a gauge transformation of $\widehat{K}$. In particular, if $\widehat{G}$ is isomorphic to a trivial central extension $G \times \mathrm{U}(1)$, then any two trivializations are related by an element of $\mathrm{Hom}(G, \mathrm{U}(1))$.

We will label central extensions by a notation $\widehat{G}^{(\kappa)}$; the *trivial central extension* is denoted $\widehat{G}^{(0)} = G \times \mathrm{U}(1)$. The *exterior product* $\widehat{G_1 \times G_2}^{(\kappa_1+\kappa_2)}$ of two central extensions is defined by the commutative diagram,

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{U}(1) \arrow[r] \arrow[d, "="] & (\widehat{G_1}^{(\kappa_1)} \times \widehat{G_2}^{(\kappa_2)}) \times \mathrm{U}(1) \arrow[r] \arrow[d] & \widehat{G_1}^{(\kappa_1)} \times \widehat{G_2}^{(\kappa_2)} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathrm{U}(1) \arrow[r] & \widehat{G_1 \times G_2}^{(\kappa_1+\kappa_2)} \arrow[r] & G_1 \times G_2 \arrow[r] & 1 \end{tikzcd}$$

where the upper horizontal line is a trivial central extension, and the middle vertical map is the quotient map for the action

$$(w_1, w_2).(\widehat{g_1}, \widehat{g_2}, z) = (\widehat{g_1} w_1^{-1}, \widehat{g_2} w_2^{-1}, w_1 w_2 z)$$

of $U(1) \times U(1)$. If $G_1 = G_2 = G$, we define an (interior) *product* $\widehat{G}^{(\kappa_1+\kappa_2)}$ as the pull-back of the exterior under the diagonal inclusion $G \to G \times G$. Finally, the *opposite* $\widehat{G}^{(-\kappa)}$ of a central extension $\widehat{G}^{(\kappa)}$ is a quotient of the trivial central extension $\widehat{G}^{(\kappa)} \times \mathrm{U}(1)$ by the action $w.(\widehat{g}, z) = (\widehat{g}w^{-1}, zw^{-1})$. As the notation suggests, the product of $\widehat{G}^{(\kappa)}$ and $\widehat{G}^{(-\kappa)}$ is canonically isomorphic to the trivial extension.

For a compact, connected Lie group G, the group of isomorphism classes of central extensions of G by U(1) is canonically isomorphic to the cohomology group $H^3_G(\mathrm{pt}, \mathbb{Z}) = H^3(BG, \mathbb{Z})$. (Cf. [**53**] for some details.) Note that $H^3_G(\mathrm{pt}, \mathbb{Z})$ is a torsion group. Hence, if $\widehat{G}^{(\kappa)}$ is a given central extension of G, then there exists $m > 0$ such that $\widehat{G}^{(m\kappa)}$ is isomorphic to the trivial central extension. In particular, the central extension is isomorphic to the trivial central extension on the level of Lie algebras. Indeed, the choice of an invariant inner product on $\widehat{\mathfrak{g}}^{(\kappa)}$ defines a splitting of the extension, by embedding $\mathfrak{g}$ as the complement of $\mathfrak{u}(1) \subset \widehat{\mathfrak{g}}^{(\kappa)}$. The splitting is unique up to a Lie algebra morphism $\mathfrak{g} \to \mathbb{R}$, i.e. up to an element of $(\mathfrak{g}^*)^{\mathfrak{g}}$.

8.2. Twisted representations. Suppose G is a compact, connected Lie group. Given a central extension $\widehat{G}^{(\kappa)}$, we define a *κ-twisted representation of* G to be a representation of $\widehat{G}^{(\kappa)} \to \mathrm{U}(V)$ where the central circle acts with weight 1. One may think of a κ-twisted representation as a projective representation $G \to \mathrm{PU}(V)$ together with an isomorphism of $\widehat{G}^{(\kappa)}$ with the pull-back of the central extension $1 \to \mathrm{U}(1) \to \mathrm{U}(V) \to \mathrm{PU}(V) \to 1$. The isomorphism classes of κ-twisted representations form a semi-group under direct sum; let $R^{(\kappa)}(G)$ denote the Grothendieck group. The tensor product of representations defines a product

$$R^{(\kappa_1)}(G) \times R^{(\kappa_2)}(G) \to R^{(\kappa_1+\kappa_2)}(G). \tag{104}$$

In particular, $R^{(\kappa)}(G)$ is a module over $R^{(0)}(G) = R(G)$.

Fix a maximal torus $T \subset G$, and let $P_G \equiv P_T \cong \mathrm{Hom}(T, \mathrm{U}(1))$ be the weight lattice of T. We will consider P_T as a subset of $\sqrt{-1}\mathfrak{t}^*$, consisting of all $\nu \in (\mathfrak{t}^*)^{\mathbb{C}}$ such that $\langle \nu, \xi \rangle \in 2\pi\sqrt{-1}\mathbb{Z}$ whenever $\xi \in \mathfrak{t}$ is in the kernel of $\exp_T \colon \mathfrak{t} \to T$. Let $\widehat{T}^{(\kappa)} \subset \widehat{G}^{(\kappa)}$ be the maximal torus given as the pre-image of T, and define the κ-twisted weights

$$P_G^{(\kappa)} \equiv P_T^{(\kappa)} \subset \mathrm{Hom}(\widehat{T}^{(\kappa)}, \mathrm{U}(1))$$

be the homomorphisms whose restriction to the central $\mathrm{U}(1)$ is the identity. Thus, the affine lattice $P_T^{(\kappa)}$ labels the isomorphism classes of κ-twisted representations of T; in particular $R^{(\kappa)}(T) = \mathbb{Z}[P_T^{(\kappa)}]$. $P_T^{(\kappa)}$ is the affine sublattice of $P_{\widehat{G}^{(\kappa)}} = P_{\widehat{T}^{(\kappa)}}$ given as the pre-image of the generator of $P_{\mathrm{U}(1)}$. Tensor product of twisted representations of T (cf. (104) for $G = T$) gives an 'addition' map

$$P_T^{(\kappa_1)} \times P_T^{(\kappa_2)} \to P_T^{(\kappa_1+\kappa_2)}.$$

The image of the roots $\mathfrak{R} \subset \mathfrak{t}^*$ of G under the inclusion inclusion $\mathfrak{t}^* \to (\widehat{\mathfrak{t}}^{(\kappa)})^*$ are the roots of $\widehat{G}^{(\kappa)}$, and will be identified with the latter. We denote by $P_{G,+}^{(\kappa)}$ the corresponding dominant κ-twisted weights. Then $P_{G,+}^{(\kappa)}$ labels the irreducible κ-twisted representations of G, and

$$R^{(\kappa)}(G) = \mathbb{Z}[P_{G,+}^{(\kappa)}].$$

REMARK 8.1. As mentioned above, any central extension $\widehat{G}^{(\kappa)}$ of a compact connected Lie group becomes trivial on the level of Lie algebras. The restriction of a given Lie algebra splitting $\mathfrak{g} \to \mathfrak{g}^{(\kappa)}$ to $\mathfrak{t} \to \mathfrak{t}^{(\kappa)}$ dualizes to give a projection, $(\mathfrak{t}^{(\kappa)})^* \to \mathfrak{t}^*$, and embeds $P_G^{(\kappa)}$ as a subset of $\mathfrak{t}^*$ of the form

$$P_G^{(\kappa)} = P_G + \delta^{(\kappa)}.$$

The 'shift' $\delta^{(\kappa)} \in \sqrt{-1}\mathfrak{t}^*$ is defined modulo P_G; changing the trivialization by an element of $(\mathfrak{g}^*)^{\mathfrak{g}} \subset \mathfrak{t}^*$ modifies $\delta^{(\kappa)}$ accordingly.

8.3. The ρ-representation of $\mathfrak{g}$ as a twisted representation of G. Let G be compact and connected Lie group, with a given invariant inner product on its Lie algebra $\mathfrak{g}$. Define a central extension $\widehat{G}^{(\sigma)}$ as the pull-back of the central extension

$$1 \to \mathrm{U}(1) \to \mathrm{Spin}_c(\mathfrak{g}) \to \mathrm{SO}(\mathfrak{g}) \to 1$$

under the adjoint representation $G \to \mathrm{SO}(\mathfrak{g})$. Note that $\widehat{G}^{(\sigma)}$ is associated to a central extension of G by $\mathbb{Z}_2$, obtaind by pulling back the double cover $1 \to \mathbb{Z}_2 \to \mathrm{Spin}(\mathfrak{g}) \to \mathrm{SO}(\mathfrak{g}) \to 1$. It follows that this central extension is '2-torsion':

LEMMA 8.2. *Suppose $1 \to \mathbb{Z}_l \to \widetilde{G}^{(\kappa)} \to G \to 1$ is a central extension of G by $\mathbb{Z}_l$, for some $l \geq 2$, and*

$$\widehat{G}^{(\kappa)} = \widetilde{G}^{(\kappa)} \times_{\mathbb{Z}_l} \mathrm{U}(1)$$

for the natural action of $\mathbb{Z}_l$ as a subgroup of $\mathrm{U}(1)$. Then the central extension $\widehat{G}^{(l\kappa)}$ is canonically *isomorphic to the trivial central extension $\widehat{G}^{(0)}$.*

PROOF. The m-th power $\widehat{G}^{(m\kappa)}$ an associated bundle to the m-th power of $\widetilde{G}^{(m\kappa)}$. The latter is

$$\widetilde{G}^{(m\kappa)} = \widetilde{G}^{(\kappa)} \times_{\mathbb{Z}_l} \mathbb{Z}_l,$$

where $\mathbb{Z}_l$ acts on $\mathbb{Z}_l$ by the m-th power, $z.w = z^m w$. If m is a multiple of l, this is the trivial action. In particular, $\widetilde{G}^{(l\kappa)}$ is canonically trivial, hence so is $\widetilde{G}^{(l\kappa)}$. □

Realize the ρ-representation of $\mathfrak{g}$ as

$$V(\rho) \cong \gamma(U\mathfrak{g}).\mathsf{R} \subset \mathrm{Cl}(\mathfrak{g}),$$

as in §8, Section 3. If G is not simply connected, this representation need not exponeniate to the group level, and indeed ρ need not lie in P_G. But it always integrates to a representation of $\widehat{G}^{(\sigma)}$ where the central circle acts with weight 1. (Indeed, since $\xi \in \mathfrak{g}$ acts on $\gamma(U\mathfrak{g}).\mathsf{R}$ as multiplication by $\gamma(\xi)$, its exponential in the double cover of $\widetilde{G}^{(\sigma)}$ acts as multiplication by $\exp_{\mathrm{Cl}}(\gamma(\xi)) \in \mathrm{Spin}(\mathfrak{g})$.) Thus $V(\rho)$ is an irreducible σ-twisted representation of G. We denote by $\tilde{\rho} \in P_{G,+}^{(\sigma)}$ its highest weight. The splitting of the Lie algebra of $\mathrm{Spin}_c(\mathfrak{g})$ determines a splitting $\widehat{\mathfrak{g}}^{(\sigma)} = \mathfrak{g} \times \mathfrak{u}(1)$. The resulting inclusion $P_G^{(\sigma)} \hookrightarrow \mathfrak{t}^*$ (cf. Remark 8.1) takes $\tilde{\rho}$ to the half-sum of positive roots, ρ.

REMARK 8.3. Similarly, given any Clifford module S over $\mathrm{Cl}(\mathfrak{g})$, the resulting action of $\mathfrak{g}$ on S exponentiates to a σ-twisted representation of G on S, isomorphic to a direct sum of $V(\rho)$'s.

Suppose $K \subset G$ is a closed subgroup, with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$. Put $\mathfrak{p} = \mathfrak{k}^{\perp}$, and let $\widehat{K}^{(\tau)}$ be the central extension of K defined as the pull-back of the central extension

$$1 \to \mathrm{U}(1) \to \mathrm{Spin}_c(\mathfrak{p}) \to \mathrm{SO}(\mathfrak{p}) \to 1$$

under the adjoint representation $K \to \mathrm{SO}(\mathfrak{p})$. Again, this twisting is 2-torsion. Any Clifford module over $\mathrm{Cl}(\mathfrak{p})$ becomes a τ-twisted representation of K. Suppose K is a connected subgroup of maximal rank, and choose a maximal torus $T \subset K \subset G$. Let $\mathfrak{R}_{K,+} \subset \mathfrak{R}_+$ be positive roots of $K \subset G$ relative to some choice of Weyl chamber. Let $\mathsf{S}_{\mathfrak{p}}$ be the spinor module over $\mathrm{Cl}(\mathfrak{p})$, equipped with the τ-twisted representation of K as above. It need not be irreducible, in general (consider e.g. the case $K = T$). On the other hand, every irredible component for the $\mathfrak{k}$-action is also an irredicible component for the κ-twisted action of K. Let $\tilde{\rho}_{\mathfrak{p}}$ be the highest weight if the irreducible component whose underlying $\mathfrak{k}$-action has highest weight $\rho_{\mathfrak{p}}$. Similar to Example **??**, $P_K^{(\tau)}$ has a canonical embedding into $\mathfrak{t}^*$, under which $\tilde{\rho}_{\mathfrak{p}}$ goes to $\rho_{\mathfrak{p}}$. Letting $\tilde{\rho}_{\mathfrak{k}} \in P_{K,+}^{(\sigma_{\mathfrak{k}})}$ be defined similar as for $K = G$, we have

$$\tilde{\rho} = \tilde{\rho}_{\mathfrak{k}} + \tilde{\rho}_{\mathfrak{p}},$$

using the addition $P_T^{(\sigma_{\mathfrak{k}})} \times P_T^{(\tau)} \to P_T^{(\sigma)}$.

8.4. Definition of the induction map. Suppose now that G is compact and connected, K is a maximal rank subgroup, and T a maximal torus in K. Let $\widehat{K}^{(\tau)}$ be the central extension of K described in Example **??**. We will use the cubic Dirac operator to define *induction maps*

$$R^{(\kappa-\tau)}(K) \to R^{(\kappa)}(G).$$

Let M be a $(\kappa-\tau)$-twisted representation of K. Then $\mathsf{S}_{\mathfrak{p}}^* \otimes M$ is a $\mathbb{Z}_2$-graded κ-twisted representation of K. Here $\mathsf{S}_{\mathfrak{p}^*}$ is the dual of the spinor module $\mathsf{S}_{\mathfrak{p}}$.

REMARK 8.4. Recall from §3, Section 4 that the super space $\mathsf{K} = \mathrm{Hom}_{\mathrm{Cl}}(\mathsf{S}_{\mathfrak{p}}, \mathsf{S}_{\mathfrak{p}^*})$ is 1-dimensional, with parity given by $(-1)^{\frac{1}{2}\dim\mathfrak{p}} = (-1)^{|\mathfrak{R}_{\mathfrak{p},+}|}$. Hence we may replace $\mathsf{S}_{\mathfrak{p}}^* \cong \mathsf{S}_{\mathfrak{p}} \otimes \mathsf{K}$ with $\mathsf{S}_{\mathfrak{p}}$, provided the parity shift is taken into account.

The L^2-sections of the associated bundle

$$\mathsf{E} = \widehat{G}^{(\kappa)} \times_{\widehat{K}^{(\kappa)}} (\mathsf{S}_{\mathfrak{p}}^* \otimes M) \to G/K$$

are identified with the $\widehat{K}^{(\kappa)}$-invariant subspace

$$\Gamma_{L^2}(\mathsf{E}) = \big(L^2(\widehat{G}^{(\kappa)}) \otimes \mathsf{S}_{\mathfrak{p}}^* \otimes M\big)^{\widehat{K}^{(\kappa)}}, \tag{105}$$

where $\widehat{K}^{(\kappa)}$ acts on $L^2(\widehat{G}^{(\kappa)})$ by the right regular representation $(\hat{k}.f)(\hat{g}) = f(\hat{g}\hat{k})$, and on $\mathsf{S}_\mathfrak{p}^* \otimes M$ by the $\mathfrak{k}$-twisted representation as above. Since the central $\mathrm{U}(1) \subset \widehat{K}^{(\kappa)}$ acts on $\mathsf{S}_\mathfrak{p}^* \otimes M$ with weight 1, and on $L^2(\widehat{G}^{(\kappa)})$ with weight -1, it acts trivially on the tensor product. That is, the $\widehat{K}^{(\kappa)}$-action descends to an action of K, and the superscript in (105) may be replaced with K. The group $\widehat{G}^{(\kappa)}$ acts on (105) via the left regular representation on $L^2(\widehat{G}^{(\kappa)})$, $(\hat{g}_1.f)(\hat{g}) = f(\hat{g}_1^{-1}\hat{g})$. Since the left regular representation of $z \in \mathrm{U}(1)$ on $L^2(\widehat{G}^{(\kappa)})$ coincides with the right regular representation of $z^{-1} \in \mathrm{U}(1)$, we see that the central circle $\mathrm{U}(1) \subset \widehat{G}^{(\kappa)}$ acts with weight one on (105). In (105), we may replace $L^2(\widehat{G}^{(\kappa)})$ with the subspace $L^2(G)^{(\kappa)} \subset L^2(\widehat{G}^{(\kappa)}$ on which the left regular representation of U(1) has weight 1. Thus

$$\Gamma_{L^2}(\mathsf{E}) = (L^2(G)^{(\kappa)} \otimes \mathsf{S}_\mathfrak{p}^* \otimes M)^K. \tag{106}$$

From the usual direct sum decomposition of the L^2-functions on a compact Lie group one obtains that

$$L^2(G)^{(\kappa)} = \bigoplus_\pi V_\pi \otimes V_\pi^*, \tag{107}$$

where the sum is now over all level κ representations $\pi\colon \widehat{G}^{(\kappa)} \to \mathrm{Aut}(V_\pi)$. Thus

$$\Gamma_{L^2}(\mathsf{E}) = \bigoplus_\pi V_\pi \otimes \big(V_\pi^* \otimes \mathsf{S}_\mathfrak{p}^* \otimes M\big)^K,$$

where V_π^* is regarded as a $\widehat{K}^{(\kappa)}$-representation by restriction.

To obtain a finite-dimensional κ-twisted representation from the infinite-dimensional space $\Gamma_{L^2}(\mathsf{E})$, we use the relative cubic Dirac operator (cf. (89))

$$\mathcal{D}(\mathfrak{g},\mathfrak{k}) = \sum\nolimits_a^{(\mathfrak{p})} \widehat{e}_a e^a + q(\phi_\mathfrak{p}) \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}).$$

Recall that $\sum_a^{(\mathfrak{p})}$ indicates summation over a basis of $\mathfrak{p}$. Since K has maximal rank in G, we may identify $\mathfrak{p}$ with the unique K-invariant complement to $\widehat{\mathfrak{k}}^{(\kappa)}$ in $\widehat{\mathfrak{g}}^{(\kappa)}$. Thus we may also think of $\mathcal{D}(\mathfrak{g},\mathfrak{k})$ as an element of $U(\widehat{\mathfrak{g}}^{(\kappa)}) \otimes \mathrm{Cl}(\mathfrak{p})$; indeed it is just identified with $\mathcal{D}(\widehat{\mathfrak{g}}^{(\kappa)},\ \widehat{\mathfrak{k}}^{(\kappa)})$. The factor $U(\widehat{\mathfrak{g}}^{(\kappa)})$ acts on V_π^*, while $\mathrm{Cl}(\mathfrak{p})$ acts on $\mathsf{S}_\mathfrak{p}^*$; this defines an action of $\mathcal{D}(\mathfrak{g},\mathfrak{k})$ on $V_\pi^* \otimes \mathsf{S}_\mathfrak{p}^* \otimes M$. Since $\mathcal{D}(\mathfrak{g},\mathfrak{k})$ is K-invariant, it restricts to K-invariant elements, giving a collection of skew-adjoint operators

$$\mathcal{D}_M^\pi \in \mathrm{End}((V_\pi^* \otimes \mathsf{S}_\mathfrak{p}^* \otimes M)^K).$$

Tensoring with the identity operator on V_π, and summing over π one obtains a skew-adjoint (unbounded) operator $\mathcal{D}_M$ on the Hilbert space $(L^2(G)^{(\kappa)} \otimes \mathsf{S}_\mathfrak{p}^* \otimes M)^K$. Since $\mathcal{D}_M$ is equivariant, its kernel $\ker(\mathcal{D}_M)$ is a κ-twisted representation of G. We will show below that the kernel is finite-dimensional.

DEFINITION 8.5. The *Dirac induction* is the map from $(\kappa-\tau)$-twisted representations of K to $\mathbb{Z}_2$-graded κ-twisted representations of G, taking

$$M \mapsto \ker(\mathcal{D}_M).$$

It induces a map on isomorphiphism classes,

$$R^{(\kappa-\tau)}(K) \to R^{(\kappa)}(G), \quad [M] \mapsto [\ker(\mathcal{D}_M)^{\bar{0}}] - [\ker(\mathcal{D}_M)^{\bar{1}}],$$

taking $[M]$ to the equivariant index of $\mathcal{D}_M$.

8.5. The kernel of $\mathcal{D}_M$. The following Theorem gives a direct characterization of the Dirac induction map in terms of weights. For $\nu \in P^{(\kappa-\tau)}_{K,+}$, let $M(\nu)$ denote the corresponding irreducible $(\kappa-\tau)$-twisted representation of K. Similarly, for $\mu \in P^{(\kappa)}_{G,+}$ we let $N(\mu)$ denote the corresponding irreducible κ-twisted representation of G. Observe that

$$P^{(\kappa-\tau)}_K + \tilde{\rho}_{\mathfrak{k}} = P^{(\kappa)}_G + \tilde{\rho} = P^{(\kappa+\sigma)}_G.$$

The following is essentially a version of Kostant's *generalized Borel-Weil theorem* [**45, 46**], see also Landweber [**50**] and Wassermann [**61**, Section 20].

THEOREM 8.6. *Let $\nu \in P^{(\kappa-\tau)}_{K,+}$ be given. If there exists $w \in W$ (necessarily unique) such that*

$$\nu + \tilde{\rho}_{\mathfrak{k}} = w(\mu + \tilde{\rho})$$

for some $\mu \in P^{(\kappa)}_{G,+}$, then the Dirac induction takes $M(\nu)$ to $N(\mu)$, with parity change by $(-1)^{l(w)}$. The Dirac induction takes $M(\nu)$ to 0 if no such w exists.

Hence, on the level of Grothendieck groups of twisted representations, the induction $R^{(\kappa-\tau)}(K) \to R^{(\kappa)}(G)$ is given by

$$[M(\nu)] \mapsto (-1)^{l(w)}[N(\mu)]$$

if $\nu + \tilde{\rho}_{\mathfrak{k}} = w(\mu+\tilde{\rho})$ for some $w \in W$, while $[M(\nu)] \mapsto 0$ if no such W exists.

PROOF. Let $M = M(\nu)$ with $\nu \in R^{(\kappa-\tau)}_{K,+}$. Clearly,

$$\ker(\mathcal{D}_M) = \bigoplus_{\pi} V^*_\pi \otimes \ker(\mathcal{D}^\pi_M).$$

Identify

$$\begin{aligned}(V^*_\pi \otimes \mathsf{S}_{\mathfrak{p}} \otimes M(\nu))^K &\cong \mathrm{Hom}_{\widehat{K}^{(\kappa-\tau)}}(M(\nu)^*, V^*_\pi \otimes \mathsf{S}^*_{\mathfrak{p}}) \\ &= \mathrm{Hom}_{\widehat{\mathfrak{k}}^{(\kappa)}}(M(\nu)^*, V^*_\pi \otimes \mathsf{S}^*_{\mathfrak{p}}).\end{aligned}$$

Here we used that the equivariance condition relative to the action of the connected group $\widehat{K}^{(\kappa-\tau)}$ is the same as that relative to its Lie algebra, but $\widehat{\mathfrak{k}}^{(\kappa-\tau)} \equiv \widehat{\mathfrak{k}}^{(\kappa)}$ since the twist τ is torsion. As indicated in Remark 8.4, we write $\mathsf{S}_{\mathfrak{p}}^* = \mathsf{S}_{\mathfrak{p}} \otimes \mathsf{K}$. Letting $\mathcal{D}_{V_\pi^*}$ denote the action of $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$ on $V_\pi^* \otimes \mathsf{S}_{\mathfrak{p}}$, it follows that

$$\ker(\mathcal{D}_M^\pi) \cong \mathrm{Hom}_{\widehat{\mathfrak{k}}^{(\kappa)}}(M(\nu)^*, \ker(\mathcal{D}_{V_\pi^*})) \otimes \mathsf{K}.$$

Let $\mu \in (\widehat{\mathfrak{t}}^{(\kappa)})^*$ be the highest weight of V_π. Then the dual $\widehat{\mathfrak{k}}^{(\kappa)}$-representation V_π^* has highest weight $*\mu = w_0(-\mu)$ with w_0 the longest Weyl group element of W. Similarly, $M(\nu)^*$ has highest weight $*_{\mathfrak{k}}\nu \equiv w_{0,\mathfrak{k}}(-\nu)$, where $w_{0,\mathfrak{k}}$ is the longest element in $W_{\mathfrak{k}}$. By Theorem 5.1, the space

$$\mathrm{Hom}_{\widehat{\mathfrak{k}}^{(\kappa)}}(M(\nu)^*, \ker(\mathcal{D}_{V_\pi^*}))$$

is zero unless there exists $w_1 \in W$ such that $*\mu + \rho = w_1(*_{\mathfrak{k}}\nu + \rho_{\mathfrak{k}})$, and in the latter case the multiplicity is 1. The parity of this isotypical component given by the length $(-1)^{l(w_1)}$.

Since $w_0\mathfrak{R}_- = \mathfrak{R}_+$, one has $*\rho = \rho$, and hence $*\mu + \rho = *(\mu + \rho) = -w_0(\mu + \rho)$. Likewise, $*_{\mathfrak{k}}\nu + \rho_{\mathfrak{k}} = -w_{0,\mathfrak{k}}(\nu + \rho)$. We may hence re-write the condition as

$$w(\mu + \rho) = \nu + \rho_{\mathfrak{k}}$$

where $w = w_{0,\mathfrak{k}}^{-1} w_1\, w_0$. Here the condition is written in terms of Lie algebra weights; if we are using Lie group weights for the central extensions of K resp. G, the same condition reads $w(\mu + \tilde{\rho}) = \nu + \tilde{\rho}_{\mathfrak{k}}$. Since $l(w_0) = |\mathfrak{R}_+|$ and $l(w_{0,\mathfrak{k}}) = |\mathfrak{R}_{\mathfrak{k},+}|$, we have

$$(-1)^{l(w)} = (-1)^{l(w_1) + |\mathfrak{R}_{\mathfrak{p},+}|}.$$

The line K has parity $(-1)^{|\mathfrak{R}_{\mathfrak{p},+}|}$. Hence $\ker(\mathcal{D}_M^\pi)$ has parity $(-1)^{l(w)}$. We conclude that $\ker(\mathcal{D}_M)$ is isomorphic to $N(\mu)$ if $w(\mu + \tilde{\rho}) = \nu + \tilde{\rho}_{\mathfrak{k}}$ for some $w \in W$, and the parity of $\ker(\mathcal{D}_M)$ is given by the parity of $l(w)$. □

REMARK 8.7. Let us briefly compare the Dirac induction with *holomorphic induction* $R(K) \to R(G)$. Let $\mathfrak{p} = \mathfrak{k}^\perp$ carry the complex structure defined by the set $\mathfrak{R}_{\mathfrak{p},+}$ of positive roots. In general, this complex structure is only T-invariant. *Assume* that it is actually K-invariant, hence defining a complex structure on G/K. Given a K-representation M, the associated bundle $G \times_K M$ is a holomorphic vector bundle. The space of differential forms on G/K with values in this bundle, i.e. the sections of

$$G \times_K (M \times \wedge \mathfrak{p}_-),$$

carries the Dolbeault-Dirac operator $\bar{\partial}\!\!\!/_M$. One defines the holomorphic induction map by

$$[M] \mapsto [\ker(\bar{\partial}\!\!\!/_M)].$$

In terms of weights, it is given by

$$M(\nu) \mapsto (-1)^{l(w)} N(\mu)$$

provided $\nu = w(\mu + \rho) - \rho$ for some $w \in W$, and zero otherwise. (If G is not simply connected, ρ need not be in the weight lattice P_G. But $\rho - w\rho$ is a linear combination of roots with integer coefficients, and hence is always in P_G.) Similar to the discussion of Dirac induction, the computation of $\ker(\partial\!\!\!/_M)$ may be reduced to an algebraic Dirac operator on spaces

$$V \otimes \wedge \mathfrak{p}_-,$$

where V is an irreducible $\mathfrak{g}$-representation (viewed as a $\mathfrak{k}$-representation by restriction). This program was carried out in Kostant's classical paper [**43**], over 35 years before [**45**]. A detailed comparison of the two induction procedures can be found in [**46**]

CHAPTER 9

$\mathcal{D}(\mathfrak{g},\mathfrak{k})$ as a geometric Dirac operator

In the last Section we alluded to interpretations of the element $\mathcal{D}(\mathfrak{g},\mathfrak{k})$ as a geometric Dirac operator over a homogeneous space G/K. We will now discuss this interpretation in more detail.

1. Differential operators on homogeneous spaces

We begin with a general discussion of differential operators on homogeneous spaces G/K, where G is a connected Lie group and K is a closed subgroup. We denote by $\mathfrak{g},\mathfrak{k}$ the Lie algebras of G,K.

PROPOSITION 1.1. [**34**, p. 285] *Suppose $\mathfrak{k}\subset\mathfrak{g}$ admits a K-invariant complement. Then the natural map* $\mathrm{DO}(G)^{G\times K}\to\mathrm{DO}(G/K)^G$ *is onto, and defines an isomorphism*

$$\mathrm{DO}(G/K)^G=U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K\cong(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k})^K.$$

PROOF. Let $\mathfrak{p}$ be a K-invariant complement to $\mathfrak{k}$. Then

$$U(\mathfrak{g})=U(\mathfrak{g})\mathfrak{k}\oplus S(\mathfrak{p})$$

by the PBW theorem. In fact this decomposition is K-equivariant, hence it restricts to a decomposition $U(\mathfrak{g})^K=(U(\mathfrak{g})\mathfrak{k})^K\oplus S(\mathfrak{p})^K$. It follows that $U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K=(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k})^K=S(\mathfrak{p})^K$. We want to identify this space with $\mathrm{DO}(G/K)$. Under the identification of $U(\mathfrak{g})$ with left-invariant differential operators on G, the algebra $U(\mathfrak{g})^K$ corresponds to the $G\times K$-invariant differential operators, where K acts from the right. By identifying functions on G/K with right-K-invariant functions on G, one obtains a map $U(\mathfrak{g})^K\to\mathrm{DO}(G/K)^G$. Elements in $U(\mathfrak{g})\mathfrak{k}$ vanish on right-K-invariant functions (since the left-invariant vector fields generated by elements of $\mathfrak{k}$ generate right translations). We therefore obtain an algebra homomorphism

$$S(\mathfrak{p})^K\to\mathrm{DO}(G/K)^G. \tag{108}$$

To show that it is an isomorphism, it suffices to check that the associated graded map

$$S(\mathfrak{p})^K\to\mathrm{Gr}(\mathrm{DO}(G/K)^G) \tag{109}$$

is an isomorphism. We have $\mathfrak{p}\cong T_{eK}(G/K)$, so $T(G/K)=G\times_K\mathfrak{p}$. The principal symbol of a differential operator of degree r on G/K is an element of

$$\Gamma(G/K,G\times_K S^r(\mathfrak{p}))^G\cong S^r(\mathfrak{p})^K$$

(where the isomorphism evaluates a G-invariant section at the identity coset). This defines an *injective* graded algebra homomorphism

$$\mathrm{Gr}(\mathrm{DO}(G/K)^G) \hookrightarrow \mathrm{Gr}(\mathrm{DO}(G/K))^G \cong S(\mathfrak{p})^K.$$

Its composition with (109) is the identity on $S(\mathfrak{p})^K$. Thus (109) is an isomorphism, and hence so is (108). □

REMARK 1.2. For a simple example where $\mathfrak{k}$ does not admits a K-invariant complement, take $G = \mathrm{SL}(2,\mathbb{R})$ the real matrices of determinant 1, and K the upper triangular matrices with 1's on the diagonal. The Lie algebra of $\mathfrak{sl}(2,\mathbb{R})$ consists of matrices of trace 0, and has a basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{110}$$

Thus e is a generator of $\mathfrak{k}$. We have

$$\mathrm{ad}_e(h) = -2e, \quad \mathrm{ad}_e(e) = 0, \quad \mathrm{ad}_e(f) = h. \tag{111}$$

Since ad_e is nilpotent (indeed $(\mathrm{ad}_e)^3 = 0$), it must have a non-trivial kernel on any invariant subspace of $\mathfrak{g}$. But $\ker(\mathrm{ad}_e) = \mathfrak{k}$, proving that $\mathfrak{k}$ cannot have an invariant complement. The map $S(\mathfrak{g})^K \to S(\mathfrak{g}/\mathfrak{k})^K$ is non-surjective already in degree 1. Indeed, $S^1(\mathfrak{g})^K = \mathfrak{g}^K = \mathfrak{k}$ maps to 0, but $S^1(\mathfrak{g}/\mathfrak{k})^K$ is 1-dimensional, with generator the image $\bar{h}$ of h in $\mathfrak{g}/\mathfrak{k}$. The G-invariant vector field on G/K defined by $\bar{h}$ provides an example of an invariant differential operators on G/K that does not lift to $G\times K$-invariant differential operators on G.

The Proposition generalizes to differential operators acting on vector bundles. For any vector bundle $E \to M$, let $\mathrm{DO}(M,E)$ denote the algebra of differential operators on M acting on sections of E. The principal symbol of differential operators identifies the associated graded algebra with $\Gamma(S(TM)\otimes \mathrm{End}(E))$,

PROPOSITION 1.3. *Suppose $\mathfrak{k}$ admits a K-invariant complement $\mathfrak{p}$ in $\mathfrak{g}$. For any K-module V, there is an isomorphism of filtered algebras*

$$\mathrm{DO}(G/K, G\times_K V) \to (U(\mathfrak{g})\otimes \mathrm{End}(V))^K/\mathcal{I}\cap (U(\mathfrak{g})\otimes \mathrm{End}(V))^K.$$

Here $\mathcal{I}$ is the left ideal in $U(\mathfrak{g})\otimes \mathrm{End}(V)$ generated by the diagonal embedding $\mathfrak{k} \hookrightarrow \mathfrak{g}\otimes \mathrm{End}(V)$, $\xi \mapsto \tau(\xi) = \xi\otimes 1 + 1\otimes \pi(\xi)$.

PROOF. The algebra $(U(\mathfrak{g})\otimes \mathrm{End}(V))^K$ is identified with the $G\times K$-invariant differential operators $\mathrm{DO}(G, G\times V)^{G\times K}$ on the trivial bundle $G\times V$. It acts as differential operators on $G\times_K V$ using the identification $\Gamma(G\times_K V) = C^\infty(G,V)^K$. Elements in the image of the diagonal embedding $\mathfrak{k} \to \mathfrak{g}\otimes \mathrm{End}(V)$ annihilate all invariant sections, hence the same is true for all elements in the left ideal $\mathcal{I}$. We claim that there is a K-invariant direct sum decomposition

$$U\mathfrak{g}\otimes \mathrm{End}(V) = \mathcal{I}\oplus (S\mathfrak{p}\otimes \mathrm{End}(V)), \tag{112}$$

using the embedding $S\mathfrak{p} \to U\mathfrak{g}$ by symmetrization. To see this, give $U\mathfrak{g} \otimes \mathrm{End}(V)$ the filtration defined by the filtration of $U\mathfrak{g}$, and let $\mathcal{I}$ and $\mathcal{F} = (S\mathfrak{p} \otimes \mathrm{End}(V))$ carry the induced filtrations. From $U(\mathfrak{g}) = U(\mathfrak{g})\mathfrak{k} \oplus S\mathfrak{p}$ we have

$$U(\mathfrak{g})_{(r)} \otimes \mathrm{End}(V) = \mathcal{I}'_{(r)} \oplus \mathcal{F}_{(r)} \tag{113}$$

with $\mathcal{I}' = U(\mathfrak{g})\mathfrak{k} \otimes \mathrm{End}(V)$. But

$$\mathcal{I}_{(r)} = \mathcal{I}'_{(r)} \mod U(\mathfrak{g})_{(r-1)} \otimes \mathrm{End}(V),$$

hence an inductive argument deduces (112) from (113). The rest of the proof is parallel to that for $V = \mathbb{R}$. ☐

2. Geometric Dirac operators

Let M be a manifold with a pseudo-Riemannian metric B of signature n, m. Let $\mathrm{Cl}(TM)$ be the corresponding Clifford module. A connection ∇ on TM is called a *metric connection* if $\nabla B = 0$, where

$$(\nabla_X B)(Y, Z) = X\, B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

Any two metric connections differ by some tensor $S \in \Omega^1(M, \mathfrak{o}(TM))$. In a local trivialization $TM|_U = U \times \mathbb{R}^{n,m}$ (carrying B to the constant metric of signature n, m), the exterior differential defines a 'trivial' metric connection, and the given connection is of the form

$$\nabla|_U = \mathrm{d} + A_U$$

for a local connection 1-form $A_U \in \Omega^1(U, \mathfrak{o}(TM))$. By the Levi-Civita theorem, the pseudo-Riemannian manifold (M, B) carries a unique metric connection of vanishing *torsion* $T_\nabla = 0$, where T_∇ is the tensor

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

We will denote the Levi-Civita connection by ∇^{met}. Given any metric connection ∇ such that the 3-tensor

$$T_\nabla(X, Y, Z) = B(T_\nabla(X, Y), Z)$$

is skew-symmetric in X, Y, Z, the torsion free part is again a metric connection, hence it coincides with the Levi-Civita connection. That is,

$$\nabla^{met}_X(Y) = \nabla_X(Y) - \tfrac{1}{2} T_\nabla(X, Y).$$

The metric connection may be viewed as a principal connection on the orthogonal frame bundle of TM, hence it defines connections on all associated bundles, i.e. all bundles associated to some representation $\pi\colon \mathrm{O}(n, m) \to \mathrm{End}(V)$.

In particular, one obtains connections on the Clifford bundle $\mathrm{Cl}(TM)$ and on the exterior algebra bundle. In local trivializations, one again has $\nabla|_U = \mathrm{d} - \pi(A_U)$ where the letter π is also used for the infinitesimal representation $\mathfrak{o}(n, m) \to \mathrm{End}(V)$.

Suppose now that $\mathcal{E} \to M$ is a bundle of $\mathrm{Cl}(TM)$-modules. The Clifford action will be denoted $\varrho\colon \mathrm{Cl}(TM) \to \mathrm{End}(\mathcal{E})$. A connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ is called a *Clifford connection* if

$$\nabla^{\mathcal{E}}(\varrho(x)\phi) = \varrho(\nabla(x))\phi + \varrho(x)\nabla^{\mathcal{E}}\phi$$

for all $x \in \Gamma(\mathrm{Cl}(TM))$, $\phi \in \Gamma(\mathcal{E})$. Here ∇ is the given metric connection on TM, and the same notation is used for its extension to $\mathrm{Cl}(TM)$. Any two Clifford connections differ by a section of $\mathrm{End}_{\mathrm{Cl}(TM)}(\mathcal{E})$. Given a Clifford connection, one defines the corresponding *Dirac operator*

$$\partial\!\!\!/^{\mathcal{E}}\colon \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$$

as a composition

$$\Gamma(\mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} \Gamma(T^*M \otimes \mathcal{E}) \xrightarrow{\varrho} \Gamma(\mathcal{E}).$$

(using B to identify T^*M with TM). In terms of a local frame e_a of TM, with B-dual frame e^a,

$$\partial\!\!\!/^{\mathcal{E}} = \sum_a \varrho(e^a)\,\nabla^{\mathcal{E}}_{e_a}.$$

3. Dirac operators over G

Let G be a Lie group, with an invariant quadratic form B on its Lie algebra. Let TG carry the corresponding bi-invariant pseudo-Riemannian metric, still denote by B.

PROPOSITION 3.1. *There is a unique left-invariant connection ∇^{nat} on TG, with the property that*

$$\nabla^{nat}_X(Y) = 0, \qquad X, Y \in \mathfrak{X}(G)^L \tag{114}$$

on left-invariant vector fields. Its values on right-invariant vector fields are

$$\nabla^{nat}_X(Y) = [X, Y], \quad X, Y \in \mathfrak{X}(G)^R. \tag{115}$$

In fact (114) *holds whenever Y is left-invariant, and* (115) *holds whenever X is right-invariant. The torsion of ∇^{nat} is given by $T(X,Y) = -[X,Y]$ if both X, Y are left-invariant, and by $T(X,Y) = [X,Y]$ of both X, Y are right-invariant. The connection ∇^{nat} is a metric connection, i.e. $\nabla^{nat}B = 0$, and its geodesics are the left-translates (or equivalently the right-translates) of the 1-parameter subgroups of G.*

Equation in (114) says that ∇^{nat} is the 'trivial' connection relative to the left-trivialization of TG.

PROOF. For arbitrary $X, Y \in \mathfrak{X}(G)$ define ∇^{nat}_X by

$$\iota(\nabla^{nat}_X(Y))\theta^L = \mathcal{L}_X \iota_Y \theta^L = \iota_{[X,Y]}\theta^L + \iota_Y \mathcal{L}_X \theta^L.$$

It is easy to check that this formula defines a connection, i.e. ∇^{nat}_X is tensorial in X and $\nabla^{nat}_X(fY) = X(f)Y + f\nabla^L_X(Y)$. If Y is left-invariant, then $\iota_Y\theta^L$

is constant and hence $L_X\iota_Y\theta^L = 0$, hence $\nabla^{nat}_X(Y) = 0$. If X is right-invariant, then $\mathcal{L}_X\theta^L = 0$ and we read off $\nabla^{nat}_X(Y) = [X,Y]$. The two expressions for the torsion follow directly from the definition $T(X,Y) = \nabla^{nat}_X(Y) - \nabla^{nat}_Y(X) - [X,Y]$, for the special case that X,Y are both left-invariant or both right-invariant. To show $\nabla^{nat}B = 0$, we show that

$$(\nabla^{nat}_X B)(Y,Z) = XB(Y,Z) - B(\nabla^{nat}_X Y, Z) - B(Y, \nabla^{nat}_X Z)$$

vanishes on left-invariant vector fields. But $B(Y,Z)$ is constant if Y,Z are left-invariant, while $\nabla^{nat}_X Y$, $\nabla^{nat}_X Z$ vanish if X is left-invariant. A curve $\gamma(s)$ is a geodesic for ∇^{nat} if and only if the velocity vector field $\dot\gamma$ (a vector field along the curve γ, i.e. a section of $\gamma^*(TG)$) satisfies $\nabla^{nat}_{\dot\gamma}\dot\gamma = 0$. Here the left hand side may be calculated by choosing a vector field X for which γ is an integral curve, and computing $\nabla_X X|_{\gamma(s)}$. If $\gamma(s) = \exp(s\xi)$ is a 1-parameter subgroup, one may take $X = \xi^L$, and it follows immediately that γ is a geodesic. By the uniqueness theorem, *all* geodesics starting at e are of this form. By left-invariance of the metric, the geodesics starting at $a \in G$ are left translates by a of geodesics starting at e. □

Any connection $\nabla_X(Y)$ can be turned into a torsion-free connection by subtracting half its torsion, $\overline{\nabla}_X(Y) = \nabla_X(Y) = -\frac{1}{2}T(X,Y)$. The geodesics of $\nabla, \overline{\nabla}$ coincide. In the case of ∇^{nat}, since $B(T(X,Y),Z)$ is skew-symmetric in X,Y,Z the torsion-free part coincides with the Levi-Civita connection:

$$\nabla^{met}_X(Y) = \nabla^{nat}_X(Y) - \tfrac{1}{2}T(X,Y)$$

Thus

$$\nabla^{met}_X(Y) = \tfrac{1}{2}[X,Y]$$

if X,Y are left-invariant. More generally, let us introduce a family of left-invariant connections by putting

$$\nabla^t_X(Y) = t[X,Y]$$

on left-invariant vector fields. Its torsion is $T^t(X,Y) = (2t-1)[X,Y]$. Thus ∇^0 is the natural connection, and $\nabla^{1/2}$ is the Levi-Civita connection. In terms of left-trivialization of the tangent bundle, we have

$$\nabla^t_X = \mathcal{L}_X + t\,\mathrm{ad}_{\tilde X}$$

Suppose now that V is a $\mathrm{Cl}(\mathfrak{g})$-module. Then $\mathcal{E} = G \times V$ is a left-invariant $\mathrm{Cl}(TG) = G \times \mathrm{Cl}(T\mathfrak{g})$-module (i.e. Clifford action commutes with left translation). The formula

$$\nabla^{\mathcal{E},t}_X = \mathcal{L}_X + t\varrho(\gamma_{\tilde X})$$

defines a connection on $\mathcal{E}$.

LEMMA 3.2. *The connection $\nabla^{\mathcal{E},t}$ is a Clifford connection (relative to the given connection ∇^t).*

PROOF. We check, for $\gamma \in \Gamma(\mathcal{E})$ (viewed as a function $\widetilde{\gamma} \in C^\infty(G, V)$),

$$\begin{aligned}\widetilde{\nabla_X^{\mathcal{E},t} \varrho(Y)\gamma} &= (\mathcal{L}_X + t\varrho(\gamma_{\tilde{X}}))\varrho(\tilde{Y})\widetilde{\gamma} \\ &= \varrho(\tilde{Y})\widetilde{\nabla_X^{\mathcal{E},t}\gamma} + \varrho(\mathcal{L}_X \tilde{Y} + t[\tilde{X}, \tilde{Y}])\widetilde{\gamma} \\ &= \varrho(\tilde{Y})\widetilde{\nabla_X^{\mathcal{E},t}\gamma} + \varrho(\widetilde{\nabla_X^t Y})\widetilde{\gamma}\end{aligned}$$

□

The resulting Dirac operator on $\mathcal{E}$ is

$$\partial\!\!\!/^t = \sum_a \varrho(e^a)(\mathcal{L}(e_a) + t\varrho(\gamma_{e_a})).$$

Comparing with the formula $\mathcal{D} = \sum_a e^a \widehat{e_a} + \frac{1}{3}\sum_a e^a \gamma(e_a)$ for the algebraic Dirac operator we see that the latter corresponds to the value $t = \frac{1}{3}$. In terms of ∇^{nat} and ∇^{met}, this is the convex linear combination $\nabla^{1/3} = \frac{1}{3}\nabla^{nat} + \frac{2}{3}\nabla^{met}$ We have shown:

THEOREM 3.3. *Suppose V is a $\mathrm{Cl}(V)$-module. The image of $\mathcal{D}$ under the representation*

$$U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}) \to \mathrm{DO}(G,\, G \times V)$$

as left-invariant differential operators is the geometric Dirac operator defined by the left-invariant connection $\frac{1}{3}\nabla^{nat} + \frac{2}{3}\nabla^{met}$.

4. Dirac operators over G/K

Let us quickly recall the construction of connections on vector bundles, associated to principal connections $\theta \in \Omega^1(P, \mathfrak{k})^K$ on principal K-bundles $P \to M$. Given such a vector bundle $P \times_K V$, there is a natural isomorphism

$$\Gamma(P \times_K V) \cong C^\infty(P, V)^K, \quad \sigma \mapsto \widetilde{\sigma}$$

defined by the pull-back of sections. This extends to differential forms with values in $P \times_K V$,

$$\Omega(M,\, P \times_K V) \to \Omega(P, V)_{K-\mathrm{bas}}, \quad \gamma \mapsto \tilde{\gamma};$$

the unique extension as an $\Omega(M) \cong \Omega(P)_{K-\mathrm{bas}}$-module homomorphism. The linear connection

$$\nabla\colon \Omega(M, P \times_K V) \to \Omega^{\cdot+1}(M, P \times_K V)$$

corresponding to the principal connection θ is given by the formula,

$$\widetilde{\nabla\gamma} = (\mathrm{d} + \theta)\widetilde{\gamma},$$

where the $\mathfrak{k}$-part of $\theta \in \Omega^1(M, \mathfrak{k})^K$ acts by the infinitesimal action on V. Equivalently,

$$\widetilde{\nabla_X \gamma} = \iota(\mathrm{hor}(X))\mathrm{d}\widetilde{\gamma},$$

where $\mathrm{hor}(X) \in \mathfrak{X}(P)$ is the horizontal lift of $X \in \mathfrak{X}(M)$ with respect to θ.

Consider a homogeneous space G/K. Its tangent space at the identity is $T_{eK}(G/K) = \mathfrak{g}/\mathfrak{k}$, hence by equivariance $T(G/K) = G\times_K(\mathfrak{g}/\mathfrak{k})$. The choice of a G-invariant principal connection on $G \to G/K$ is equivalent to the choice of a K-equivariant splitting of the sequence $0 \to \mathfrak{k} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{k} \to 0$, i.e. to the choice of a K-invariant complement $\mathfrak{p} \subset \mathfrak{g}$. Letting $\mathrm{pr}_{\mathfrak{k}}\colon \mathfrak{g} \to \mathfrak{k}$ denote the projection along $\mathfrak{p}$, its connection 1-form is $\theta = \mathrm{pr}_{\mathfrak{k}}\,\theta^L$, and for the curvature $F^\theta = \mathrm{d}\theta + \frac{1}{2}[\theta,\theta]$ one finds (using $\mathrm{d}\theta^L + \frac{1}{2}[\theta^L,\theta^L] = 0$, and decomposing into $\mathfrak{k}$-parts and $\mathfrak{p}$-parts)

$$F^\theta = -\tfrac{1}{2}\,\mathrm{pr}_k[\mathrm{pr}_{\mathfrak{p}}\,\theta^L, \mathrm{pr}_{\mathfrak{p}}\,\theta^L].$$

As above, the connection θ induces G-invariant linear connections on all associated vector bundles $G\times_K V$. As a special case, we may take $V = \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$. The resulting connection on $T(G/K) = G\times_K\mathfrak{p}$ is called the *natural connection* on $T(G/K)$; it will be denoted ∇^{nat}. In the case $K = \{e\}$, this is the connection ∇^{nat} defined by left-trivialization of the tangent bundle. The function $\widetilde{X}$ defined by a vector field $X \in \mathfrak{X}(G/K) \cong \Gamma(G\times_K\mathfrak{p})$ is related to the horizontal lift by $\widetilde{X} = \mathrm{hor}(X)\theta^L$.

Lemma 4.1. *The identification $\mathfrak{X}(G/K) \cong C^\infty(G,\mathfrak{p})^K,\ X \to \widetilde{X}$ takes the Lie bracket to*

$$\widetilde{[X,Y]} = \mathrm{hor}(X)\tilde{Y} - \mathrm{hor}(Y)\tilde{X} + \mathrm{pr}_{\mathfrak{p}}[\tilde{X},\tilde{Y}].$$

Proof. This follows from the definition:

$$\begin{aligned}
\widetilde{[X,Y]} &= \iota(\mathrm{hor}([X,Y]))\theta^L \\
&= \iota([\mathrm{hor}(X),\mathrm{hor}(Y)])\theta^L + F^\theta(\mathrm{hor}(X),\mathrm{hor}(Y)) \\
&= \mathrm{hor}(X)\tilde{Y} - \mathrm{hor}(Y)\tilde{X} + [\tilde{X},\tilde{Y}] + F^\theta(\mathrm{hor}(X),\mathrm{hor}(Y)) \\
&= \mathrm{hor}(X)\tilde{Y} - \mathrm{hor}(Y)\tilde{X} + \mathrm{pr}_{\mathfrak{p}}[\tilde{X},\tilde{Y}].
\end{aligned}$$

□

Lemma 4.2. *The formula for the connection reads,*

$$\widetilde{\nabla^{nat}_X Y} = \mathrm{hor}(X)\widetilde{Y}.$$

Its torsion is given by $\widetilde{T(X,Y)} = -\,\mathrm{pr}_{\mathfrak{p}}[\widetilde{X},\widetilde{Y}]$.

Proof. We have

$$\widetilde{\nabla^{nat}_X Y} - \widetilde{\nabla^{nat}_Y X} - \widetilde{[X,Y]} = \mathrm{hor}(X)\widetilde{Y} - \mathrm{hor}(Y)\widetilde{X} - \widetilde{[X,Y]} = -\,\mathrm{pr}_{\mathfrak{p}}[\widetilde{X},\widetilde{Y}].\ \square$$

As before, we introduce a family of connections

$$\nabla^t_X(Y) = \nabla^{nat}_X(Y) - tT(X,Y)$$

so that $t = \frac{1}{2}$ corresponds is a torsion-free connection. Suppose now that $\mathfrak{g}$ is a quadratic Lie algebra, $\mathfrak{k}$ a quadratic Lie subalgebra, and $\mathfrak{p} = \mathfrak{k}^\perp$.

The pseudo-Riemannian metric on G descends to a left-invariant pseudo-Riemannian metric on G/K, and one observes that the connections ∇^t are all metric connections. In particular, $\nabla^{1/2}$ is the Levi-Civita connection.

For $\xi \in \mathfrak{p}$, let $\mathrm{ad}^{\mathfrak{p}}_\xi \colon \mathfrak{p} \to \mathfrak{p}$ be the skew-adjoint map $\zeta \mapsto \mathrm{pr}_{\mathfrak{p}}[\xi, \zeta]$. Then

$$\widetilde{\nabla^t_X(Y)} = \mathrm{hor}(X)\widetilde{Y} + t\, \mathrm{ad}^{\mathfrak{p}}_{\tilde{X}}(\tilde{Y}).$$

Let $\gamma^{\mathfrak{p}}(\xi) \in \mathrm{Cl}(\mathfrak{p})$ be the Clifford algebra element corresponding to $\mathrm{ad}^{\mathfrak{p}}_\xi$. [...] If V is a K-equivariant $\mathrm{Cl}(\mathfrak{p})$-module, define a connection on $\mathcal{E} = G \times_K V$ by

$$\widetilde{\nabla^{\mathcal{E},t}_X \sigma} = \mathrm{hor}(X)\widetilde{\sigma} + t\varrho(\gamma^{\mathfrak{p}}_{\tilde{X}})\widetilde{\sigma}.$$

As before, one directly checks that this is a Clifford connection: Observe that the map $\mathfrak{p} \to TG,\ \xi \mapsto \xi^L$ gives an isometric isomorphism with the horizontal spaces. Hence the Dirac operator is $\widetilde{\partial\!\!\!/^t \sigma} = \sum'_a \varrho(e^a) \nabla^{\mathcal{E},t}_{e_a} \widetilde{\sigma}$, where the summation is over a basis e_a of $\mathfrak{p}$, with dual basis e^a. That is,

$$\widetilde{\partial\!\!\!/^t \sigma} = {\sum_a}' \varrho(e^a)(\mathcal{L}(e^L_a) + t\varrho(\gamma^{\mathfrak{p}}(e_a)))\widetilde{\sigma}$$

Arguing as in the case $K = \{e\}$ we have:

THEOREM 4.3. *The image of* $\mathcal{D}(\mathfrak{g}, \mathfrak{k})$ *under the map*

$$(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^K \to \mathrm{DO}(G/K,\ G \times_K V)$$

is the geometric Dirac operator for the connection $\nabla = \frac{1}{3}\nabla^{nat} + \frac{2}{3}\nabla^{met}$.

REMARK 4.4. Dirac operators on homogeneous spaces are discussed in a number of references, such as Ikeda [**37**]. The geometric Dirac operator $\partial\!\!\!/^{1/3}$ was first considered by Slebarski [**58, 59**], who also observed that the square of this operator is given by a simple formula. The identification of $\partial\!\!\!/^{1/3}$ with the algebraic Dirac operator was noticed by S. Goette [**30**], and proved in detail by I. Agricola [**2**].

APPENDIX A

Graded and filtered super spaces

1. Super vector spaces

A *super vector space* is a vector space E equipped with a $\mathbb{Z}_2$-grading, $E = E^{\bar{0}} \oplus E^{\bar{1}}$. Elements in $E^{\bar{0}}$ will be called even, elements of $E^{\bar{1}}$ are called odd. We will denote by $|v| \in \{\bar{0}, \bar{1}\}$ the parity of homogeneous elements $v \in E$. (Whenever this notation is used, it is implicitly assumed that v is homogeneous.) A *morphism of super vector spaces* $\phi\colon E \to F$ is a linear map preserving $\mathbb{Z}_2$-gradings. We will denote by $\mathrm{Hom}(E, F)$ the space of *all* linear maps $E \to F$, not necessarily preserving $\mathbb{Z}_2$-gradings. It is itself a super vector space, with

$$\mathrm{Hom}(E,F)^{\bar{0}} = \mathrm{Hom}(E^{\bar{0}}, F^{\bar{0}}) \oplus \mathrm{Hom}(E^{\bar{1}}, F^{\bar{1}}),$$
$$\mathrm{Hom}(E,F)^{\bar{1}} = \mathrm{Hom}(E^{\bar{0}}, F^{\bar{1}}) \oplus \mathrm{Hom}(E^{\bar{1}}, F^{\bar{0}}).$$

The space of morphisms of super vector spaces $E \to F$ is thus the even subspace $\mathrm{Hom}(E,F)^{\bar{0}}$. Direct sums and tensor products of super vector spaces are just the usual tensor products of vector spaces, with $\mathbb{Z}_2$-gradings

$$(E \oplus F)^{\bar{0}} = E^{\bar{0}} \oplus F^{\bar{0}}, \qquad (E \oplus F)^{\bar{1}} = E^{\bar{1}} \oplus F^{\bar{1}},$$

respectively

$$(E \otimes F)^{\bar{0}} = (E^{\bar{0}} \otimes F^{\bar{0}}) \oplus (E^{\bar{1}} \otimes F^{\bar{1}}),$$
$$(E \otimes F)^{\bar{1}} = (E^{\bar{0}} \otimes F^{\bar{1}}) \oplus (E^{\bar{1}} \otimes F^{\bar{0}}).$$

If E is a super vector space, and $n \in \mathbb{Z}$, we denote by $E[n]$ the same vector space with $\mathbb{Z}_2$-grading shifted by $n \mod 2$.

The '*super sign convention*' decrees that the interchange of any two odd objects results in a minus sign. We will take the categorical viewpoint, advocated in [**24**], that the super-sign convention is build into the choice of commutativity isomorphism for the tensor product:

$$\mathcal{T}\colon\ E \otimes F \mapsto F \otimes E,\quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.$$

The category of super vector spaces is then a *tensor category*, with direct sums, tensor products and a commutativity isomorphism as defined above. One may then define super algebras, super Lie algebras, super coalgebras etc. as the algebra objects, Lie algebra objects, coalgebra objects etc. in this tensor category. Similarly, various constructions with these objects are naturally defined in terms of 'categorical constructions'.

For example, a *super Lie algebra* is a super space $\mathfrak{g}$ together with a bracket $= [\cdot,\cdot]\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ satisfying the following axioms:

$$[\cdot,\cdot]\circ\mathcal{T} = -[\cdot,\cdot]$$

(skew symmetry) and

$$[\cdot,\cdot]\circ(\mathrm{id}\otimes[\cdot,\cdot]) = [\cdot,\cdot]\circ([\cdot,\cdot]\otimes\mathrm{id}) + [\cdot,\cdot]\circ(\mathrm{id}\otimes[\cdot,\cdot])\circ(\mathcal{T}\otimes\mathrm{id})$$

(Jacobi identity). On homogeneous elements, the two conditions read

$$[u,v] = -(-1)^{|u||v|}[v,u],$$

$$[u,[v,w]] = [[u,v],w] + (-1)^{|u||v|}[v,[u,w]].$$

Super algebras are $\mathbb{Z}_2$-graded algebras $\mathcal{A}$, such that the multiplication map $m_{\mathcal{A}}\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ preserves $\mathbb{Z}_2$-gradings. The super sign convention makes its appearance once we define the tensor product $\mathcal{A}\otimes\mathcal{B}$ of two such algebras. By definition the multiplication of the tensor algebra is

$$m_{\mathcal{A}\otimes\mathcal{B}} = (m_{\mathcal{A}}\otimes m_{\mathcal{B}})\circ(\mathrm{id}\otimes\mathcal{T}\otimes\mathrm{id}),$$

a composition of maps $\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{A}\otimes\mathcal{B}\to\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{B}\to\mathcal{A}\otimes\mathcal{B}$. Writing $(m_{\mathcal{A}}\otimes m_{\mathcal{B}})((x\otimes y)\otimes(x'\otimes y')) = (x\otimes y)(x'\otimes y')$ the multiplication map is

$$(x\otimes y)(x'\otimes y') = (-1)^{|x'||y|}\; xx'\otimes yy'.$$

The sign convention also shows up if one writes out the definition of the commutator $[\cdot,\cdot] = m_{\mathcal{A}} - m_{\mathcal{A}}\otimes\mathcal{T}$ for a super algebra $\mathcal{A}$:

$$[x,y] = xy - (-1)^{|x||y|}yx.$$

This bracket makes $\mathcal{A}$ into a super Lie algebra. The center $\mathrm{Cent}(\mathcal{A})$ of the super algebra $\mathcal{A}$ is the collection of elements x such that $[x,\mathcal{A}]=0$. The super algebra $\mathcal{A}$ is called commutative if $\mathrm{Cent}(\mathcal{A})=\mathcal{A}$, i.e. $[\mathcal{A},\mathcal{A}]=0$. A *trace* on a super algebra is a morphism $\mathrm{tr}\colon\mathcal{A}\to\mathbb{K}$ that vanishes on $[\mathcal{A},\mathcal{A}]\subset\mathcal{A}$. An endomorphism $D\in\mathrm{End}(\mathcal{A})$ is called a *derivation* of the super algebra if

$$D(xy) = (Dx)y + (-1)^{|D||x|}x(Dy)$$

for all homogeneous elements x,y. The space $\mathrm{Der}(\mathcal{A})$ of such derivations is a super Lie subalgebra of $\mathrm{End}(\mathcal{A})$. Some basic properties of derivations of a super algebra are,

(1) Any $D\in\mathrm{Der}(\mathcal{A})$ vanishes on scalars $\mathbb{K}\subset\mathcal{A}$. This is immediate from the definition, applied to $x=y=1$.
(2) Derivations are determined by their values on algebra generators.
(3) $\mathrm{Der}(\mathcal{A})$ is a left module under $\mathrm{Cent}(\mathcal{A})$.
(4) The map $x\mapsto[x,\cdot]$ defines a morphism of super Lie algebras $\mathcal{A}\to\mathrm{Der}(\mathcal{A})$. Derivations of this type are called *inner*.

One similarly defines tensor products, the center, and derivations of super Lie algebras.

REMARK 1.1. Super algebras can be viewed as ordinary algebras by forgetting the $\mathbb{Z}_2$-grading. To avoid misunderstandings, one sometimes refers to commutators, the center, traces, and derivations of a super algebra as super commutators, the super center, super traces, and super derivations. (In some of the older literature, the terms 'anti-commutator' and 'anti-derivation' are also used.)

For any super vector space E, the tensor algebra $T(E)$ is a super algebra, characterized by the universal property:

PROPOSITION 1.2. *For any super algebra $\mathcal{A}$ and morphism of super vector spaces $E \to \mathcal{A}$, there is a unique extension to a morphism of super algebras $T(E) \to \mathcal{A}$.*

The symmetric algebra $S(E)$ is the quotient of the tensor algebra by the two-sided ideal generated by all $v \otimes w - (-1)^{|v||w|} w \otimes v$ for homogeneous elements $v, w \in E$. Its universal property reads,

PROPOSITION 1.3. *For any commutative super algebra $\mathcal{A}$ and morphism of super vector spaces $E \to \mathcal{A}$, there is a unique extension to a morphism of super algebras $S(E) \to \mathcal{A}$.*

2. Graded super vector spaces

A graded vector space is a vector space equipped with a $\mathbb{Z}$-grading $E = \bigoplus_{k\in\mathbb{Z}} E^k$. The degree of a homogeneous element v is denoted $|v| \in \mathbb{Z}$. A morphism of graded vector spaces is a degree preserving linear map. The direct sum of two graded vector spaces E, F is graded as $(E \oplus F)^k = E^k \oplus F^k$, while the tensor product is graded as

$$(E \otimes F)^k = \bigoplus_{i\in\mathbb{Z}} E^i \otimes F^{k-i}.$$

One can make graded vector spaces into a tensor category in two ways. Taking the commutativity isomorphism $E \otimes F \to F \otimes E$ to be $v \otimes w \mapsto w \otimes v$, one obtains what we will call the category of *graded vector spaces.* Taking the isomorphism to be $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$, one obtains what we will refer to as *graded super vector spaces.* In the second case, the $\mathbb{Z}_2$-grading is just the mod 2 reduction of the $\mathbb{Z}$-grading.

The algebra objects, Lie algebra objects, and so on in the category of graded vector spaces will be called *graded algebras*, *graded Lie algebras*, and so on, while those in the category of graded super vector spaces will be called *graded super algebras*, *graded super Lie algebras*, and so on. For instance, a graded (super) Lie algebra is a graded (super) vector space, which is also a (super) Lie algebra, and with the property $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$.

EXAMPLE 2.1. The exterior algebra $\wedge(V) = \bigoplus_k \wedge^k(V)$ over a vector space V is an example of a commutative graded super algebra. Under direct sum, $\wedge(V \oplus W) = \wedge(V) \otimes \wedge(W)$ as graded super algebras (but not as ordinary algebras). On the other hand, $S(V)$ is a commutative graded algebra. One has $S(V \oplus W) = S(V) \otimes S(W)$ as graded algebras (but not as super algebras).

It is often convenient to regard graded spaces as graded super spaces, by *degree doubling*. For example, a symmetric algebra $S(V) = \bigoplus_k S^k(V)$ over a vector space V is commutative graded algebra. It becomes a commutative graded superalgebra for the doubled grading

$$S(V)^{2k} = S^k(V),\ S^{2k+1}(V) = 0.$$

Given a graded vector space E, and any $n \in \mathbb{Z}$, one defines new graded vector space $E[n]$ by degree shift:

$$(E[n])^k = E^{n+k}.$$

Thus, if $v \in E$ has degree k, then its degree in $E[n] \cong E$ is $k - n$. For a graded super vector space, this operation changes the $\mathbb{Z}_2$-grading by $(-1)^n$. Note also that $E[n] = E \otimes \mathbb{K}[n]$.

If E, F are graded (super) vector spaces, define $\mathrm{Hom}^k(E, F)$ as the set of linear maps $\phi\colon E \to F$ of degree k, i.e. such that $\phi(E^i) \subset F^{k+i}$. Equivalently, $\mathrm{Hom}^k(E, F)$ consists of morphisms of graded vector spaces $E \to F[k]$. One has

$$\bigoplus_k \mathrm{Hom}^k(E, F) \subset \mathrm{Hom}(E, F) \subset \prod_k \mathrm{Hom}^k(E, F).$$

In general, each of the two inclusions can be strict.

Example 2.2. Suppose E, F are graded (super) vector spaces. Then

$$E^* = \mathrm{Hom}(E, \mathbb{K}) = \prod_k \mathrm{Hom}^k(E, \mathbb{K}) = \prod_k (E^{-k})^*$$

while

$$F \cong \mathrm{Hom}(\mathbb{K}, F) = \bigoplus_k \mathrm{Hom}^k(\mathbb{K}, F) = \bigoplus_k F^k.$$

If $F = E$ we write $\mathrm{End}^k(E) = \mathrm{Hom}^k(E, E)$. Then $\bigoplus_k \mathrm{End}^k(E)$ is a graded Lie algebra under commutator. Similarly, if E is a graded super vector space, then $\bigoplus_k \mathrm{End}^k(E)$ becomes a graded super Lie algebra.

Suppose $\mathcal{A}$ is a graded algebra, and let $\mathrm{Der}(\mathcal{A})$ be the Lie algebra of derivations. The elements of

$$\mathrm{Der}^k(\mathcal{A}) = \mathrm{Der}(\mathcal{A}) \cap \mathrm{End}^k(\mathcal{A})$$

are called *derivations of degree k of the graded algebra $\mathcal{A}$*. The direct sum $\bigoplus_k \mathrm{Der}^k(\mathcal{A})$ becomes a graded Lie subalgebra of $\bigoplus_k \mathrm{End}^k(\mathcal{A})$. In a similar way, one defines *derivations of degree k of a graded super algebra*, by taking $\mathrm{Der}(\mathcal{A})$ to be the derivations as a super algebra.

If E is a graded super vector space, then the tensor algebra $T(E)$ and the symmetric algebra $S(E)$ acquire the structure of graded super algebras, in such a way that the inclusion map $E \to T(E)$ resp. $E \to S(E)$ is a morphism of graded super vector spaces. These *internal gradings* are not to be confused with the *external gradings* $T(E) = \bigoplus_{k\geq 0} T^k(E)$ resp. $S(E) = \bigoplus_{k\geq 0} S^k(E)$. Sometimes, we also consider the *total gradings*, given by the internal grading plus twice the external grading. Then $S(E), T(E)$ are also graded superalgebra relative to the total grading.

3. Filtered super vector spaces

A *filtered vector space* is a vector space E together with a sequence of subspaces $E^{(k)},\ k \in \mathbb{Z}$ such that $E^{(k)} \subset E^{(k+1)}$ and

$$\bigcap_k E^{(k)} = 0, \ \ \bigcup_k E^{(k)} = E.$$

A *morphism of filtered vector spaces* is a linear map $\phi\colon E \to F$ taking $E^{(k)}$ to $F^{(k)}$, for all k. Direct sums of filtered vector spaces are filtered in the obvious way, while tensor products are filtered as

$$(E \otimes F)^{(k)} = \bigoplus_i E^{(i)} \otimes F^{(k-i)}.$$

Filtered vector spaces form a tensor category, hence we can speak of filtered algebras, filtered Lie algebras and so on by requiring that the relevant structure maps should be morphisms. A typical example of a filtered algebra is the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra. If E, F are filtered vector spaces, we define $\mathrm{Hom}^{(k)}(E, F)$ to be the space of linear maps $E \to F$ raising filtration degree by k. Note that the union over these spaces is usually smaller than $\mathrm{Hom}(E, F)$, since a general linear map $E \to F$ need not have any finite filtration degree.

We put $\mathrm{End}^{(k)}(E) = \mathrm{Hom}^{(k)}(E, E)$. If $\mathcal{A}$ is a filtered algebra, we define $\mathrm{Der}^{(k)}(\mathcal{A}) = \mathrm{Der}(\mathcal{A}) \cap \mathrm{End}^{(k)}(\mathcal{A})$.

Suppose E is a filtered vector space. A linear map $E \to \mathbb{K}$ has filtration degree k if and only if it takes $E^{(l)}$ to 0 for all $l < -k$. That is,

$$(E^*)^{(k)} = \mathrm{ann}(E^{(-k-1)}).$$

If the filtration of E is bounded below in the sense that $E^{(l)} = 0$ for some l, then any element of E^* has finite filtration degree, and hence the $(E^*)^{(k)}$ define a filtration of E^*.

Suppose E is a filtered vector space. The *associated graded vector space* $\mathrm{gr}(E)$ is defined as the direct sum over $\mathrm{gr}^k(E) = E^{(k)}/E^{(k-1)}$. A morphism ϕ of filtered vector spaces induces a morphism $\mathrm{gr}(\phi)$ of associated graded spaces. In this way, gr becomes a functor from the tensor category of filtered vector spaces to the tensor category of graded vector spaces. The associated graded object to a filtered algebra is a graded algebra, the associated graded object to a filtered Lie algebra is a graded Lie algebras, and so on. If E carries a filtration, which is bounded in the sense that $E^{(l)} = 0$ and $E^{(m)} = E$ for some l, m, then the filtration on E^* is bounded, and $\mathrm{gr}(E^*) = \mathrm{gr}(E)^*$.

Any graded vector space $E = \bigoplus_k E^k$ can be regarded as a filtered vector space, by putting $E^{(k)} = \bigoplus_{i\leq k} E^i$. In this case, $\mathrm{gr}(E) \cong E$.

A *filtered super vector space* [**48**] is a super vector space E, equipped with a filtration $E^{(k)}$ by super subspaces, such that

$$(E^{(k)})^{\bar{0}} = (E^{(k+1)})^{\bar{0}}, \quad \text{for } k \text{ even},$$
$$(E^{(k)})^{\bar{1}} = (E^{(k+1)})^{\bar{1}}, \quad \text{for } k \text{ odd}.$$

Equivalently, the $\mathbb{Z}_2$-grading on the associated graded space $\mathrm{gr}(E)$ is the mod 2 reduction of the $\mathbb{Z}$-grading, making $\mathrm{gr}(E)$ into a graded super vector space. If E is a filtered super vector space with a bounded filtration, then E^* is again a filtered super vector space. An example of a filtered super space is the Clifford algebra $\mathrm{Cl}(V;B)$ of a vector space V with bilinear form B. The filtered super vector spaces form a tensor algebra, hence there are notions of filtered super algebras, filtered super Lie algebras, a space $\mathrm{Der}^{(k)}(\mathcal{A})$ of degree k derivations of a filtered super algebra, and so on.

If E is a filtered super vector space, then the tensor algebra $T(E)$ and the symmetric algebra $S(E)$ acquire the structure of filtered super algebras, in such a way that the inclusion map $E \to T(E)$ resp. $E \to S(E)$ is a morphism of filtered super vector spaces. We will refer to this as the *internal filtration.* Sometimes we also consider the *total filtration*, obtained by adding twice the external filtration degree. The total filtration is such that the map $E[-2] \to T(E)$ resp. resp. $E[-2] \to S(E)$ are filtration preserving.

APPENDIX B

Reductive Lie algebras

Throughout this section, $\mathbb{K}$ denotes a field of characteristic zero, and all Lie algebra are taken finite-dimensional. We soon specialize to the case $\mathbb{K} = \mathbb{C}$. Standard references for the material below are Bourbaki [**12, 13, 14**] and Humphreys [**36**].

1. Definitions and basic properties

A Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ is called *simple* if it non-abelian and has no ideals other than $\{0\}$ and $\mathfrak{g}$. A Lie algebra is called *semi-simple* if it is a direct sum of simple Lie algebras. A Lie algebra is *reductive* if it is the direct sum of a semi-simple and an abelian Lie algebra. These conditions can be expressed in a number of equivalent ways. Most importantly, a Lie algebra is semi-simple if and only if the *Killing form*

$$B_{\mathrm{Kil}}(\xi,\zeta) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_\xi\, \mathrm{ad}_\zeta)$$

is non-degenerate. Cf. [**12**, I.§6.1]. Hence, reductive Lie algebras are in particular quadratic Lie algebras (one may take the Killing form on the semi-simple part $[\mathfrak{g},\mathfrak{g}]$ and an arbitrary non-degenerate symmetric bilinear form on the center). The existence of a finite-dimensional representation $\pi\colon \mathfrak{g} \to \mathrm{End}(V)$ such that the bilinear form $B_V(\xi,\zeta) = \mathrm{tr}_V(\pi(\xi)\pi(\zeta))$ is non-degenerate is one characterization of reductive Lie algebras. Cf. [**12**, I.§6.4].

Examples 1.1. (1) The 3-dimensional Lie algebra $\mathfrak{g}$ with basis e, f, h and bracket relations

$$[e,f] = h,\quad [h,e] = 2e,\quad [h,f] = -2f \tag{116}$$

is semi-simple. It is isomorphic to the Lie algebra $\mathfrak{sl}(2,\mathbb{K})$ of trace-free 2×2-matrices under the identification

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(2) The Lie algebra $\mathfrak{gl}(n,\mathbb{K}) = \mathrm{Mat}_n(\mathbb{K})$ of $n\times n$-matrices is reductive, and its subalgebra $\mathfrak{sl}(n,\mathbb{K})$ of trace-free matrices is semi-simple.
(3) For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, the Lie algebra $\mathfrak{so}(n,\mathbb{K})$ is semi-simple for $n \geq 3$.
(4) The Lie algebra $\mathfrak{g}$ of any compact real Lie group G is reductive. It is semi-simple if and only if the center of G is finite.

According to *Weyl's Theorem*, any finite-dimensional representation V of a semi-simple Lie algebra is a direct sum $V = \bigoplus_i V_i$ of irreducible ones. (Cf. [**12**, I.§6.1].) This property of complete reducibility does not hold for reductive Lie algebras, in general. (Take e.g. $\mathfrak{g} = \mathbb{K}$, acting on $V = \mathbb{K}^2$ as strictly upper triangular 2×2-matrices.)

A Lie algebra $\mathfrak{g}$ over $\mathbb{K} = \mathbb{R}$ is called *compact* if it admits an invariant symmetric bilinear form that is positive definite. $\mathfrak{g}$ is compact if and only if there is a compact Lie group G having $\mathfrak{g}$ as its Lie algebra. Cf. [**14**, IX.§1.3].

Suppose $\mathfrak{g}$ is a Lie algebra over $\mathbb{K} = \mathbb{C}$. A Lie algebra $\mathfrak{g}_\mathbb{R}$ is called a *real form* of $\mathfrak{g}$ if $\mathfrak{g} = \mathfrak{g}_\mathbb{R} \otimes \mathbb{C}$. A complex Lie algebra $\mathfrak{g}$ is reductive if and only if it admits a compact real form $\mathfrak{g}_\mathbb{R}$. Cf. [**14**, IX.§3.3]. Any two real forms of a complex reductive Lie algebra $\mathfrak{g}$ are related by an automorphism of $\mathfrak{g}$.

2. Cartan subalgebras

A Lie algebra $\mathfrak{g}$ is *nilpotent* if the series of ideals

$$\mathfrak{g},\ [\mathfrak{g},\mathfrak{g}],\ [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]],\ [\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]],\ \ldots$$

is eventually zero. A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra that is equal to its own normalizer, i.e. $\mathrm{ad}_\xi(\mathfrak{h}) \subseteq \mathfrak{h} \Rightarrow \xi \in \mathfrak{h}$.

Examples 2.1.
(1) Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$. Then the subalgebra $\mathfrak{h}$ of diagonal matrices is a Cartan subalgebra.
(2) For any $\xi \in \mathfrak{g}$, let $\mathfrak{g}^0(\xi)$ be the generalized eigenspace of ad_ξ for the eigenvalue 0. (I.e. $\zeta \in \mathfrak{g}^0(\xi)$ if and only if $\mathrm{ad}_\xi^n \zeta = 0$ for n sufficiently large.) An element $\xi \in \mathfrak{g}$ is called *regular* if and only if $\dim \mathfrak{g}^0(\xi)$ takes on its smallest possible value. The latter is called the *rank* of $\mathfrak{g}$. If ξ is regular, then $\mathfrak{g}^\xi$ is a Cartan subalgebra. Cf. [**14**, VII.§2.3]. Thus, the rank of $\mathfrak{g}$ is the dimension of a Cartan subalgebra.
(3) As a special case, consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{K})$ with its standard basis e, f, h. Then $\mathfrak{g}$ has rank 1. The element h is regular, hence it spans a Cartan subalgebra. The sum $e+f$ is another example of a regular element.
(4) If $\mathfrak{g}$ is complex reductive, and $\mathfrak{t}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$ is a maximal Abelian subalgebra of a compact real form of $\mathfrak{g}$, then $\mathfrak{t} = \mathfrak{t}_\mathbb{R} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$.

Cartan subalgebras of semi-simple Lie algebras are always commutative, and consist of semi-simple elements. Cf. [**14**, VII.§2.4]. If $\mathfrak{g}$ is semi-simple and $\mathbb{K} = \mathbb{C}$, any two Cartan subalgebras are conjugate in $\mathfrak{g}$. The same holds for compact Lie algebras over $\mathbb{K} = \mathbb{R}$. In this case, the Cartan subalgebras are exactly the maximal Abelian subalgebras. (This is not true in general: E.g. for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{K})$, the span of f is maximal abelian but is not a Cartan subalgebra.)

3. Representation theory of $\mathfrak{sl}(2,\mathbb{C})$.

We need some facts from the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. Let e, f, h be the standard basis as above.

THEOREM 3.1. *Up to isomorphism, there is a unique $k+1$-dimensional irreducible representation $V(k)$ of $\mathfrak{sl}(2,\mathbb{C})$, for any $k \geq 0$. It admits a basis $v_0, \ldots, v_k$ such that for all $j = 0, \ldots, k$,*

$$\pi(f)v_j = (j+1)v_{j+1}, \quad \pi(h)v_j = (k-2j)v_j, \quad \pi(e)v_j = (k-j+1)v_{j-1}$$

with the convention $v_{k+1} = 0,\ v_{-1} = 0$.

PROOF. It is straightforward to verify that these formulas define a representation of $\mathfrak{sl}(2,\mathbb{C})$. Since $\pi(e)^{k+1} = 0$, the operator $\pi(e)$ has a non-zero kernel on every invariant subspace of $V(k)$. But $\ker(\pi(e))$ is spanned by v_0. It follows that every invariant subspace contains v_0, and hence also contains the vectors $v_j = \frac{1}{j!}\pi(f)^j v_0$. This shows that $V(k)$ is irreducible.

Suppose conversely that $\pi\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(V)$ is any finite-dimensional irreducible representation. Let $v \in V$ be an eigenvector of $\pi(h)$, with eigenvalue s. If $\pi(e)v \neq 0$, then $\pi(e)v$ is an eigenvector of $\pi(h)$ with eigenvalue $s+2$, by the calculation

$$\pi(h)\pi(e)v = \pi([h,e])v - \pi(e)\pi(h)v = (s+2)\pi(e)v.$$

Since $\dim V < \infty$, the sequence of vectors $v,\ \pi(e)v,\ \pi(e)^2 v, \ldots$ is eventually zero. Take v_0 to be the last non-zero element in this sequence. Then $\pi(e)v = 0,\ \ \pi(h)v = s_0 v$ for some $s_0 \in \mathbb{C}$. Define

$$v_j = \frac{1}{j!}\pi(f)^j v_0, \quad j = 0, 1, \ldots.$$

Arguing as above, v_j is an eigenvector of $\pi(h)$ with eigenvalue $s_0 - 2j$, provided that it is non-zero. Hence, the sequence of v_j's is eventually 0, and the non-zero v_j are linearly independent. Let $k \geq 0$ be defined by $v_k \neq 0,\ v_{k+1} = 0$. By construction, the span of $v_0, \ldots, v_k$ is invariant under $\pi(f)$ and under $\pi(h)$. Using

$$\pi(e)v_{j+1} = \frac{1}{j+1}\pi(e)\pi(f)v_j = \frac{1}{j+1}(\pi(h) + \pi(f)\pi(e))v_j$$

and induction on j one proves that $\pi(e)v_{j+1} = (s_0 - j)v_j$. Taking $j = k$ this identity shows $s_0 = k$. Hence, $v_0, \ldots, v_k$ span a copy of $V(k)$. Since V is irreducible, this span coincides with all of V. $\square$

REMARK 3.2. The representation $V(k)$ of dimension $k+1$ admits a concrete realization as the k-th symmetric power of the defining representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^2$.

We will often use the following simple consequence of these formulas:

COROLLARY 3.3. *Let* $\pi\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(V)$ *be a finite-dimensional* $\mathfrak{sl}(2,\mathbb{C})$*-representation. The the operator* $\pi(h)$ *on* V *is diagonalizable, and its eigenvalues are integers. Moreover, letting* $V_r = \ker(\pi(h) - r)$*, we have:*

$$r > 0 \Rightarrow \pi(f)\colon V_r \to V_{r-2} \text{ is injective}$$
$$r < 0 \Rightarrow \pi(e)\colon V_r \to V_{r+2} \text{ is injective.}$$

PROOF. The statements hold true for all irreducible components, hence also for their direct sum. □

4. Roots

Assume for the remainder of this Appendix that $\mathbb{K} = \mathbb{C}$. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$, with a given compact real form $\mathfrak{g}_{\mathbb{R}}$, and let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra obtained by complexification of a maximal Abelian subalgebra $\mathfrak{t}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$. The choice of $\mathfrak{g}_{\mathbb{R}}$ determines a complex conjugation map $*\colon \mathfrak{g} \to \mathfrak{g}$, with $\mathfrak{g}_{\mathbb{R}}$ as its fixed point set. We will also fix an invariant positive definite symmetric bilinear form B on $\mathfrak{g}_{\mathbb{R}}$, and use the same notation for its complexification to a bilinear form on $\mathfrak{g}$. The resulting bilinear form on $\mathfrak{g}^*$ will be denoted by B^*.

For any $\alpha \in \mathfrak{t}^*$, define the subspace

$$\mathfrak{g}_\alpha = \{\zeta \in \mathfrak{g} |\ \xi \in \mathfrak{t} \Rightarrow \mathrm{ad}_\xi \zeta = \langle \alpha, \xi\rangle \zeta\ \}.$$

Then $\mathfrak{g}$ is a direct sum over the non-zero subspaces $\mathfrak{g}_\alpha$. Elementary properties of these subspaces are

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta},$$
$$\mathfrak{g}_\alpha^* = \mathfrak{g}_{-\alpha}$$
$$\mathfrak{g}_0 = \mathfrak{t}$$

Furthermore, $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\alpha + \beta \neq 0$, while $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ are non-singularly paired for all $\alpha \neq 0$.

A non-zero element $\alpha \in \mathfrak{t}^*$ is called a *root* of $\mathfrak{g}$ if $\mathfrak{g}_\alpha \neq 0$; the corresponding subspace $\mathfrak{g}_\alpha$ is called a *root space*. The set of roots is denoted $\mathfrak{R}$. Thus

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\mathfrak{R}} \mathfrak{g}_\alpha.$$

EXAMPLE 4.1. Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ with its standard basis e, f, h, and with $\mathfrak{t} = \mathrm{span}(h)$. Then $\alpha(h) = \pm 2$ (resp. $\alpha(h) = -2$) defines a root, with corresponding root space $\mathfrak{g}_\alpha = \mathrm{span}(e)$ (resp. $\mathrm{span}(f)$).

PROPOSITION 4.2. *For all* $\alpha \in \mathfrak{R}$*, the space* $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ *is 1-dimensional, and is spanned by* $B^\sharp(\alpha)$*. More precisely, if* $e \in \mathfrak{g}_\alpha$ *and* $f \in \mathfrak{g}_{-\alpha}$*, then*

$$[e, f] = B(e, f)\ B^\sharp(\alpha).$$

PROOF. For $e \in \mathfrak{g}_\alpha,\ f \in \mathfrak{g}_{-\alpha}$ and $h \in \mathfrak{t}$ we have

$$B(\mathrm{ad}(e)f, h) = -B(f, \mathrm{ad}(e)h) = B(e, f)\langle \alpha, h\rangle.$$

□

Definition 4.3. For any root $\alpha \in \mathfrak{R}$, the *co-root* $\alpha^\vee \in \mathfrak{t}$ is the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\langle \alpha, \alpha^\vee \rangle = 2$. The set of co-roots is denoted $\mathfrak{R}^\vee$.

In terms of the bilinear form, one has

$$\alpha^\vee = 2\frac{B^\sharp(\alpha)}{B^*(\alpha,\alpha)}.$$

Proposition 4.4. *Let $e_\alpha \in \mathfrak{g}_\alpha$ with normalization $B(e_\alpha, e_\alpha^*) = \frac{2}{B(\alpha^\vee,\alpha^\vee)}$, and put $f_\alpha = e_\alpha^*$, $h_\alpha = \alpha^\vee$. Then $e_\alpha, f_\alpha, h_\alpha$ are the standard basis for an $\mathfrak{sl}(2,\mathbb{C})$-subalgebra $\mathfrak{sl}_\alpha \subset \mathfrak{g}$.*

Proof. We have $[h_\alpha, e_\alpha] = \langle \alpha, h_\alpha \rangle e_\alpha = 2e_\alpha$ and similarly $[h_\alpha, f_\alpha] = -2f_\alpha$. Furthermore,

$$[e_\alpha, f_\alpha] = B(e_\alpha, f_\alpha)B^\sharp(\alpha) = \alpha^\vee = h_\alpha.$$

□

The representation theory of the $\mathfrak{sl}(2,\mathbb{C})$ subalgebras $\mathfrak{sl}_\alpha$ implies the basic properties of root systems.

Proposition 4.5. *Let $\alpha \in \mathfrak{R}$ be a root. Then:*

(1) $\dim(\mathfrak{g}_\alpha) = 1$.
(2) *If $\beta \in \mathfrak{R}$ is a multiple of α, then $\beta = \pm\alpha$.*
(3) *Let $\beta \in \mathfrak{R}$, with $\beta \neq \pm\alpha$. Then*
$$\langle \beta, \alpha^\vee \rangle < 0 \Rightarrow \alpha + \beta \in \mathfrak{R}.$$
(4) *(Root strings.) Given $\beta \in \mathfrak{R}$, with $\beta \neq \pm\alpha$, there exist integers $q, p \geq 0$ such that for any integer r, $\beta + r\alpha \in \mathfrak{R}$ if and only if $-q \leq r \leq p$. These integers satisfy*
$$q - p = \langle \beta, \alpha^\vee \rangle.$$
The direct sum $\bigoplus_{r=-q}^{p} \mathfrak{g}_{\beta+r\alpha}$ is an irreducible $\mathfrak{sl}_\alpha$-representation of dimension $p+q+1$.
(5) *If $\alpha, \beta, \alpha+\beta$ are all roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

Proof. We will regard $\mathfrak{g}$ as an $\mathfrak{sl}_\alpha$-representation, by restricting the adjoint representation of $\mathfrak{g}$. That is, the standard basis elements act as $\mathrm{ad}(h_\alpha), \mathrm{ad}(e_\alpha), \mathrm{ad}(f_\alpha)$.

(1) Since $\mathrm{ad}(h_\alpha)$ acts on $\mathfrak{g}_{-\alpha}$ as a non-zero scalar -2, Corollary 3.3 shows that the map $\mathrm{ad}(e_\alpha)\colon \mathfrak{g}_{-\alpha} \to \mathfrak{g}_0$ is injective. On the other hand, Proposition 4.2 shows that its range is 1-dimensional. Hence $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 1$.

(2) We may assume that $t\alpha$ is not a root for $|t| < 1$. We will show that it is not a root for $|t| > 1$. Suppose on the contrary that $t\alpha$ is a root for some $t > 1$, and take the smallest such t. The operator $\mathrm{ad}(h_\alpha)$ acts on $\mathfrak{g}_{t\alpha}$ as a positive scalar $2t > 0$. By Corollary 3.3, it follows that $\mathrm{ad}(f_\alpha)\colon \mathfrak{g}_{t\alpha} \to \mathfrak{g}_{(t-1)\alpha}$ is injective. Since $t > 1$, and since there are no smaller multiples of α that are roots, other than α itself, this implies that $t = 2$, and $\mathrm{ad}(f_\alpha)\colon \mathfrak{g}_{2\alpha} \to \mathfrak{g}_\alpha$

is injective. But this is impossible, since $\mathfrak{g} = \mathfrak{sl}_\alpha \oplus \mathfrak{sl}_\alpha^\perp$ is an $\mathfrak{sl}_\alpha$-invariant decomposition with $\mathfrak{g}_\alpha \subset \mathfrak{sl}_\alpha$, $\mathfrak{g}_{2\alpha} \subset \mathfrak{sl}_\alpha^\perp$.

(3) Suppose α, β are distinct roots with $\langle \beta, \alpha^\vee \rangle < 0$. Since $\mathrm{ad}(h_\alpha)$ acts on $\mathfrak{g}_\beta$ as a negative scalar $\langle \beta, \alpha^\vee \rangle < 0$, Corollary 3.3 shows that $\mathrm{ad}(e_\alpha)\colon \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$ is injective. Since $\alpha + \beta \neq 0$ it is thus an isomorphism. In particular, $\mathfrak{g}_{\alpha+\beta}$ is non-zero.

(4) If $\mathfrak{g}_{\beta+r\alpha} \neq 0$ then $\mathrm{ad}(h_\alpha)$ acts on it as a scalar $\langle \beta, \alpha^\vee \rangle + 2r$. Let q, p be the largest integers such that $\mathfrak{g}_{\beta+p\alpha} \neq 0$, respectively $\mathfrak{g}_{\beta-q\alpha} \neq 0$. Then $\mathfrak{g}_{\beta+p\alpha} \subset \ker(\mathrm{ad}(e_\alpha))$. Consequently,

$$V = \bigoplus_{j \geq 0} \mathrm{ad}^j(f_\alpha)\mathfrak{g}_{\beta+p\alpha}$$

is an irreducible $\mathfrak{sl}_\alpha$-representation $V \subset \mathfrak{g}$. Its dimension is $k+1$ where $k = \langle \beta, \alpha^\vee \rangle + 2p$ is the eigenvalue of $\mathrm{ad}(h_\alpha)$ on $\mathfrak{g}_{\beta+p\alpha}$. In particular, all subspaces $\mathfrak{g}_{\beta+r\alpha}$ for $-k \leq \langle \beta, \alpha^\vee \rangle + 2r \leq k$ are non-zero. By a similar argument, we see that $\mathfrak{g}_{\beta-q\alpha}$ and its images under $\mathrm{ad}^j(e_\alpha)$, $j = 0, 1, 2, \ldots$ span an irreducible representation V' of dimension $k'+1$, where $k' = 2q - \langle \beta, \alpha^\vee \rangle$. Since all root spaces $\mathfrak{g}_{\beta+r\alpha}$ with $\langle \beta, \alpha^\vee \rangle + 2r > 0$ are contained in V, we must have $V = V'$, $k = k'$. In particular, $\langle \beta, \alpha^\vee \rangle + 2p = 2q - \langle \beta, \alpha^\vee \rangle$, hence $q - p = \langle \beta, \alpha^\vee \rangle$ and $k = q + p$.

(5) follows from (4), since $\mathrm{ad}(e_\alpha)\colon \mathfrak{g}_\beta \to \mathfrak{g}_{\beta+\alpha}$ is an isomorphism if $\mathfrak{g}_\beta$, $\mathfrak{g}_{\beta+\alpha}$ are non-zero. ☐

Definition 4.6 (Lattices).
(1) The lattice

$$Q = \mathrm{span}_\mathbb{Z}\, \mathfrak{R} \subseteq \sqrt{-1}\mathfrak{t}_\mathbb{R}^*$$

spanned by the roots is called the *root lattice*.

(2) The lattice

$$Q^\vee = \mathrm{span}_\mathbb{Z}\, \mathfrak{R}^\vee \subseteq \sqrt{-1}\mathfrak{t}_\mathbb{R}$$

spanned by the co-roots is called the *co-root lattice*.

(3) The lattice

$$P = \{\mu \in \sqrt{-1}\mathfrak{t}_\mathbb{R}^* |\ \xi \in Q^\vee \Rightarrow \langle \mu, \xi \rangle \in \mathbb{Z}\}$$

dual to the co-root lattice is called the *weight lattice*. Similarly, $P^\vee = Q^* \subset \mathfrak{t}$ is called the co-weight lattice.

Part (4) of Proposition 4.5 shows in particular that $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \mathfrak{R}_+$. Hence

$$Q \subseteq P,$$

and dually $Q^\vee \subseteq P^\vee$. We will find it convenient to work with the bilinear form $(\cdot|\cdot) = -B$ on $\mathfrak{g}$, given by $(\xi_1|\xi_2) = -B(\xi_1, \xi_2)$, so that $(\cdot, \cdot)$ is positive definite on $\sqrt{-1}\mathfrak{t}_\mathbb{R}$. The same notation $(\cdot|\cdot)$ will be used for the dual bilinear form on $\mathfrak{g}^*$.

5. Simple roots

Fix $\xi_0 \in \mathfrak{t}$ with $\langle \alpha, \xi_0 \rangle \in \mathbb{R}\backslash\{0\}$ for all $\alpha \in \mathfrak{R}$. The choice of ξ_0 determines a decomposition

$$\mathfrak{R} = \mathfrak{R}_+ \cup \mathfrak{R}_-$$

into *positive roots* and *negative roots*, where $\mathfrak{R}_+$ (resp. $\mathfrak{R}_-$) consists of all roots such that $\langle \alpha, \xi_0 \rangle > 0$ (resp. < 0). Note $\alpha \in \mathfrak{R}_- \Leftrightarrow -\alpha \in \mathfrak{R}_+$, and that the sum of two positive roots (resp. of two negative roots) is again positive (resp. negative).

Definition 5.1. A positive root is called *simple* if it cannot be written as a sum of two positive roots.

Proposition 5.2 (Simple roots). *The set of simple roots has the following properties.*

(1) *The simple roots $\alpha_1, \ldots \alpha_l$ form a basis of the root lattice.*
(2) *A root $\alpha \in \mathfrak{R}$ is positive if and only if all coefficients in the expansion $\alpha = \sum_i k_i \alpha_i$ are ≥ 0. It is negative if and only if all coefficients are ≤ 0.*
(3) *One has $\langle \alpha_i, \alpha_i^\vee \rangle \leq 0$ for $i \neq j$.*

Proof. If α_i, α_j are distinct simple roots, then their difference $\alpha_i - \alpha_j$ is *not* a root. (Otherwise, either $\alpha_i = \alpha_j + (\alpha_i - \alpha_j)$ or $\alpha_j = \alpha_i + (\alpha_j - \alpha_i)$ would be a sum of two positive roots.) It follows that distinct simple roots satisfy $\langle \alpha_i, \alpha_j^\vee \rangle \leq 0$, proving (3).

We next show that the α_i are linearly independent. Indeed suppose $\sum_i k_i \alpha_i = 0$. Let

$$\mu = \sum_{k_i > 0} k_i \alpha_i = -\sum_{k_j < 0} k_j \alpha_j.$$

Taking the scalar product with itself, and using $B(\alpha_i, \alpha_j) \leq 0$ for $i \neq j$ we obtain

$$0 \leq (\mu|\mu) = -\sum_{k_i > 0,\ k_j < 0} k_i k_j (\alpha_i|\alpha_j) \leq 0.$$

Hence $\mu = 0$, which shows that all $k_i = 0$, proving (1).

We claim that any $\alpha \in \mathfrak{R}_+$ can be written in the form $\alpha = \sum k_i \alpha_i$ for some $k_i \in \mathbb{Z}_{\geq 0}$. Otherwise, let α be a counterexample with $\langle \alpha, \xi_0 \rangle$ as small as possible. Since α is not a simple root, it can be written as a sum $\alpha = \alpha' + \alpha''$ of two positive roots α', α''. Then $\langle \alpha', \xi_0 \rangle, \langle \alpha'', \xi_0 \rangle$ are both strictly smaller than $\langle \alpha, \xi_0 \rangle$. Hence, neither is a counterexample, and each can be written as a linear combination of α_i's with non-negative coefficients. Hence the same is true of α. This proves (2). □

6. The Weyl group

Associated to any root α is a reflection $w_\alpha \in \mathrm{GL}(\mathfrak{t})$ given by

$$w_\alpha(\xi) = \xi - \langle \alpha, \xi \rangle \alpha^\vee.$$

Dually, one has a reflection $w_\alpha \in \mathrm{GL}(\mathfrak{t}^*)$ given by

$$w_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha.$$

Thus $\langle w_\alpha(\mu),\ w_\alpha(\xi)\rangle = \langle \mu, \xi\rangle$ for all $\mu \in \mathfrak{t}^*,\ \xi \in \mathfrak{t}$. The reflection $w_\alpha \in \mathrm{GL}(\mathfrak{t})$ admits an extension to a Lie algebra automorphism $\mathfrak{g}$, as follows. Let $e_\alpha, h_\alpha, f_\alpha$ be the basis for $_\alpha$, as defined in Proposition 4.4.

Proposition 6.1. *The transformation*

$$\theta_\alpha = \exp(\mathrm{ad}(e_\alpha)) \exp(-\mathrm{ad}(f_\alpha)) \exp(\mathrm{ad}(e_\alpha)) \in \mathrm{GL}(\mathfrak{g})$$

is well-defined Lie algebra automorphism of $\mathfrak{g}$. It has the property $\theta_\alpha|_\mathfrak{t} = w_\alpha$, and restricts to isomorphisms $\mathfrak{g}_\beta \to g_{w_\alpha(\beta)}$ for all roots β.

Proof. Since $\mathrm{ad}(e_\alpha), \mathrm{ad}(f_\alpha)$ are nilpotent Lie algebra derivations of $\mathfrak{g}$, the exponentials are well-defined Lie algebra automorphisms of $\mathfrak{g}$. If $h \in \ker(\alpha) \subset \mathfrak{t}$, then $\mathrm{ad}(e_\alpha)h = 0 = \mathrm{ad}(f_\alpha)h$, hence $\theta_\alpha(h) = h$. On the other hand, if $h = h_\alpha$ we may replace $\mathfrak{g}$ with the $\mathfrak{sl}_\alpha$, and hence assume $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, with $e_\alpha = e,\ f_\alpha = f,\ h_\alpha = h$ given by matrices as in Example 1.1. Then $\theta_\alpha = \theta$ is the transformation

$$\theta = \mathrm{Ad}(\exp(e)\exp(-f)\exp(e))$$

using exponentials of matrices. We compute

$$\exp(e)\exp(-f)\exp(e) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

conjugation of $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ by this matrix gives $-h$.

For the second part, let $v \in \mathfrak{g}_\beta$, and calculate

$$\mathrm{ad}(h)\theta_\alpha(v) = \theta_\alpha(\mathrm{ad}(w_\alpha^{-1}h)v) = \langle \beta, w_\alpha^{-1}h\rangle \theta_\alpha(v) = \langle w_\alpha(\beta),\ h\rangle \theta_\alpha(v).$$

$\square$

Corollary 6.2. *The transformations w_α of $\mathfrak{t}^*, \mathfrak{t}$ preserve the sets $\mathfrak{R}, \mathfrak{R}^\vee$, respectively. Hence they also preserve the lattices $Q \subset P$ and $Q^\vee \subset P^\vee$, respectively.*

Proof. By the Proposition, if $\mathfrak{g}_\beta$ is non-zero then so is $\mathfrak{g}_{w_\alpha(\beta)}$. Hence w_α preserves the set of roots. Using the formula $\beta^\vee = 2(\beta|\cdot)/(\beta|\beta)$ for the co-roots, we see that w_α is just the orthogonal reflection defined by $(\cdot|\cdot)$, and that $w_\alpha(\beta^\vee) = (w_\alpha\beta)^\vee$. In particular, $\mathfrak{R}^\vee$ is w_α-invariant as well. $\square$

Definition 6.3. The subgroup $W \subset \mathrm{GL}(\mathfrak{t})$ generated by the reflections $w_\alpha, \alpha \in \mathfrak{R}$ is called the *Weyl group* of the pair $(\mathfrak{g}, \mathfrak{t})$ (or of the root system $\mathfrak{R} \subset \mathfrak{t}^*$).

We have $w_\alpha w_\beta w_\alpha = w_{\beta'}$ where $\beta' = w_\alpha(\beta)$. Hence, any $w \in W$ can be written as a product of at most $|\mathfrak{R}_+|$ reflections w_α. In particular, $|W| < \infty$.

Proposition 6.4. *The reflection $w_i = w_{\alpha_i}$ defined by a simple root α_i permutes the set $\mathfrak{R}_+ \backslash \{\alpha_i\}$.*

PROOF. Suppose $\alpha \in \mathfrak{R}_+ \backslash \{\alpha_i\}$. Write $\alpha = \sum_j k_j \alpha_j \in \mathfrak{R}_+$, so that all $k_j \geq 0$. The root

$$w_i\alpha = \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i = \sum_j k_j' \alpha_j. \tag{117}$$

has coefficients $k_j' = k_j$ for $j \neq i$. Since α is not a multiple of α_i it follows that $k_j' = k_j > 0$ for some $j \neq i$. This shows that $w_i\alpha$ is positive. □

For element $\alpha = \sum_i k_i \alpha_i$ of the root lattice, one defines its *height* as

$$\mathrm{ht}(\alpha) = \sum_i k_i.$$

The formula (117) for $w_i\alpha$ shows that if $\langle \alpha, \alpha_i^\vee \rangle > 0 \Rightarrow \mathrm{ht}(w_i\alpha) < \mathrm{ht}(\alpha)$.

PROPOSITION 6.5. *Any Weyl group element w can be written as a product of simple reflections*

$$w = w_{i_1} \ldots w_{i_r} \tag{118}$$

PROOF. It suffices to show that every w_α, $\alpha \in \mathfrak{R}_+$ can be written in this form. The proof uses induction on $k = \mathrm{ht}(\alpha)$. Since $0 < (\alpha|\alpha) = \sum_i k_i(\alpha|\alpha_i)$, there is at least one i with $(\alpha|\alpha_i) > 0$. As remarked above, this implies $\mathrm{ht}(w_i\alpha) < \mathrm{ht}(\alpha)$. We have

$$w_\alpha = w_i \, w_{\alpha'} \, w_i$$

with $\alpha' = w_i\alpha$; by induction $w_{\alpha'}$ is a product of simple reflections. □

DEFINITION 6.6. The *length* $l(w)$ of a Weyl group element $w \in W$ is the smallest number r such that w can be written in the form (118). If $r = l(w)$ the expression (118) is called *reduced*.

It is immediate that

$$l(w^{-1}) = l(w), \quad l(ww') \leq l(w) + l(w').$$

PROPOSITION 6.7. *For any Weyl group element w, and any simple root α_i, we have $l(ww_i) = l(w) + 1$ if $w\alpha_i$ is positive, and $l(ww_i) = l(w) - 1$ if $w\alpha_i$ is negative.*

PROOF. Write $w = w_{i_1} \cdots w_{i_r}$ with $r = l(w)$. Suppose $w\alpha_i$ is negative. Then there is $m \leq r$ such that

$$\beta := w_{i_m} \cdots w_{i_r}\alpha \in \mathfrak{R}_+, \quad w_{i_{m-1}} \cdots w_{i_r}\alpha \in \mathfrak{R}_-.$$

That is, w_{i_m} changes the positive root β to a negative root. The only positive root with this property is $\beta = \alpha_{i_m}$. With $u = w_{i_m} \cdots w_{i_r} \in W$ we have $\alpha_{i_m} = u\alpha_i$, thus $w_{i_m} = uw_iu^{-1}$. Multiplying from the right by u, and from the left by $w_{i_1} \cdots w_{i_{m-1}}$, we obtain

$$w_{i_1} \cdots w_{i_{m-1}} w_{i_{m+1}} \cdots w_{i_r} = ww_i.$$

This shows $l(ww_i) = l(w) - 1$. The case that $w\alpha_i$ is positive is reduced to the previous case, since $w'\alpha_i$ with $w' = ww_i$ is negative. □

For any $w \in W$, let $\mathfrak{R}_{+,w}$ be the set of positive roots that are made negative under w^{-1}. That is,

$$\mathfrak{R}_{+,w} = \mathfrak{R}_+ \cap w\mathfrak{R}_-. \tag{119}$$

By Proposition 6.4, $\mathfrak{R}_{+,w_i} = \{\alpha_i\}$.

PROPOSITION 6.8. *If $w = w_{i_1} \cdots w_{i_r}$, $r = l(w)$ is a reduced expression,*

$$\mathfrak{R}_{+,w} = \{\alpha_{i_1},\ w_{i_1}\alpha_{i_2},\ \cdots,\ w_{i_1}\cdots w_{i_{r-1}}\alpha_{i_r}\} \tag{120}$$

In particular,

$$l(w) = |\mathfrak{R}_{+,w}|. \tag{121}$$

PROOF. Consider $w' = ww_i$. Suppose a positive root α is made negative under $(w')^{-1} = w_i w^{-1}$. Since w_i changes the sign of $\pm\alpha_i$, and preserves both $\mathfrak{R}_+\backslash\{\alpha_i\}$ and $\mathfrak{R}_-\backslash\{-\alpha_i\}$, we see that If $w^{-1}\alpha$ is negative if and only if $w^{-1}\alpha \neq -\alpha_i$. That is,

$$\mathfrak{R}_{+,ww_i} = \begin{cases} \mathfrak{R}_{+,w} \cup \{w\alpha_i\}, & \text{if } w\alpha_i \in \mathfrak{R}_+ \\ \mathfrak{R}_{+,w}\backslash\{-w\alpha_i\}, & \text{if } w\alpha_i \in \mathfrak{R}_-. \end{cases}$$

(120) now follows by induction on $l(w)$, using that for a reduced expression $w = w_{i_1} \cdots w_{i_r}$, all $w_{i_1} \cdots w_{i_{k-1}}\alpha_{i_k}$ are positive. □

Let us now introduce the half-sum of positive roots

$$\rho := \tfrac{1}{2} \sum_{\alpha \in \mathfrak{R}_+} \alpha \in \tfrac{1}{2}\,\mathrm{span}_{\mathbb{Z}}(\mathfrak{R}_+).$$

LEMMA 6.9. *For all $w \in W$,*

$$\rho - w\rho = \sum_{\alpha \in \mathfrak{R}_{+,w}} \alpha.$$

PROOF. By definition $\mathfrak{R}_{+,w} = \mathfrak{R}_+ \cap w\mathfrak{R}_-$, with complement in $\mathfrak{R}_+$ given as $\mathfrak{R}'_{+,w} = \mathfrak{R}_+ \cap w\mathfrak{R}_+$. Hence $w\mathfrak{R}_+ = \mathfrak{R}'_{+,w} \cup (-\mathfrak{R}_{+,w})$, which gives

$$w\rho = \tfrac{1}{2} \sum_{\alpha \in \mathfrak{R}'_{+,w}} \alpha - \tfrac{1}{2} \sum_{\alpha \in \mathfrak{R}_{+,w}} \alpha = \rho - \sum_{\alpha \in \mathfrak{R}_{+,w}} \alpha.$$

□

7. Weyl chambers

Let $\mathfrak{t}_{\mathrm{reg}} \subset \mathfrak{t}_{\mathbb{R}}$ be the set of regular elements in $\mathfrak{t}_{\mathbb{R}}$, i.e. those elements for which $\ker(\mathrm{ad}_\xi) = \mathfrak{t}$. It is the complement of the union of root hyperplanes

$$H_\alpha = \{\xi \in \mathfrak{t}_{\mathbb{R}} |\ \frac{1}{2\pi\sqrt{-1}}\langle \alpha, \xi\rangle = 0\}, \quad \alpha \in \mathfrak{R}$$

The components of $\mathfrak{t}_{\mathrm{reg}}$ are called the *open Weyl chambers*. We will refer to the closures of the open chambers as the *closed Weyl chambers*, or simply *Weyl chambers*. We let

$$\mathfrak{t}_+ = \{\xi \in \mathfrak{t}_\mathbb{R} |\ \frac{1}{2\pi\sqrt{-1}}\langle\alpha, \xi\rangle \geq 0 \ \ \forall \alpha \in \mathfrak{R}_+\}.$$

be the *positive Weyl chamber*. The action of the Weyl group permutes the chambers.

We say that a root hyperplane H_α *separates the chambers* C, C' if for points x, x' in the interior of the chambers, $\frac{1}{2\pi\sqrt{-1}}\langle\alpha, x\rangle$ and $\frac{1}{2\pi\sqrt{-1}}\langle\alpha, x'\rangle$ have opposite signs, but $\frac{1}{2\pi\sqrt{-1}}\langle\beta, x\rangle$ and $\frac{1}{2\pi\sqrt{-1}}\langle\beta, x'\rangle$ have equal sign for all roots $\beta \neq \pm\alpha$.

PROPOSITION 7.1. *The Weyl group W acts simply transitively on the set of Weyl chambers. That is, every Weyl chamber is of the form $w\mathfrak{t}_+$ for a unique $w \in W$.*

PROOF. Since the Weyl group action preserves the set of roots, it also preserves the union of hyperplanes H_α. Hence W acts by a permutation on the set of Weyl chambers. Any two adjacent Weyl chambers are separated by some root hyperplane H_α, and the reflection w_α interchanges the two Weyl chambers. By a simple induction, one hence finds that any Weyl chamber is taken to $\mathfrak{t}_+$ by a finite number of Weyl reflections. An element $w \in W$ fixes $\mathfrak{t}_+$ only if it preserves the set $\mathfrak{R}_+$ of positive roots, if and only if $\mathfrak{R}_{+,w} = \emptyset$. But this means $l(w) = 0$, i.e. $w = \mathrm{id}$. $\square$

The length of a Weyl group element has the following interpretation in terms of the Weyl chambers.

PROPOSITION 7.2. *The length $l(w)$ is the number of root hyperplanes crossed by a line segment from the interior of $\mathfrak{t}_+$ to a point in the interior of $w\mathfrak{t}_+$.*

PROOF. Let $x \in \mathrm{int}(\mathfrak{t}_+),\ x' \in \mathrm{int}(w\mathfrak{t}_+)$. The line segment

$$x_t = (1-t)x + tx',\ t \in [0, 1]$$

meets the hyperplane $H_\alpha,\ \alpha \in \mathfrak{R}_+$ if and only if

$$\frac{1}{2\pi\sqrt{-1}}\langle\alpha, x'\rangle < 0 \Leftrightarrow \frac{1}{2\pi\sqrt{-1}}\langle w^{-1}\alpha, w^{-1}x'\rangle < 0 \Leftrightarrow w^{-1}\alpha \in \mathfrak{R}_- \Leftrightarrow \alpha \in \mathfrak{R}_{+,w}.$$

Hence the number of hyperplanes crossed equals $|\mathfrak{R}_{+,w}| = l(w)$. $\square$

8. Weights of representations

Let $\mathfrak{g}$ be a complex reductive Lie algebra, with Cartan subalgebra $\mathfrak{t}$. Let $\pi\colon \mathfrak{g} \to \mathrm{End}(V)$ be a representation on a (possibly infinite-dimensional) complex vector space V. An element $\mu \in \mathfrak{t}^*$ will be called a *weight* of V if the space

$$V_\mu = \{v \in V |\ \pi(\xi)v = \langle\mu, \xi\rangle v, \ \ \xi \in \mathfrak{t}\}$$

is non-zero. In this case, V_μ is called the weight space. The set of weights will be denoted $P(V)$. We observe that for all roots $\alpha \in \mathfrak{R}$,

$$\pi(e_\alpha)\colon V_\mu \to V_{\mu+\alpha}, \quad \pi(f_\alpha)\colon V_\mu \to V_{\mu-\alpha}.$$

Indeed, if $v \in V_\mu$ and $h \in \mathfrak{t}$ we have,

$$\pi(h)\pi(e_\alpha)v = \pi(e_\alpha)\pi(h)v + \pi([h,e_\alpha])v = \langle \mu, h\rangle \pi(e_\alpha)v + \langle \alpha, h\rangle \pi(e_\alpha)v$$

proving that $\pi(e_\alpha)v \in V_{\mu+\alpha}$.

EXAMPLES 8.1. (1) The weights of the adjoint representation of $\mathfrak{g}$ are

$$P(\mathfrak{g}) = \mathfrak{R} \cup \{0\}.$$

(2) The set of weights of the representation of $\mathfrak{g}$ by left multiplication on $U(\mathfrak{g})$ is empty. Indeed, suppose $x \in U(\mathfrak{g})$ is an element of filtration degree r, such that $hx = \langle \mu, h\rangle x$ for all $h \in \mathfrak{t}$. Then the image $y \in S^r(\mathfrak{g})$ of x satisfies $hy = 0$ for all $h \in \mathfrak{t}$, hence $y = 0$. It follows that x has filtration degree $r-1$. Proceeding by induction we find $x = 0$.

The second example illustrates that if $\dim V = \infty$, the direct sum of weight spaces may be strictly smaller than V. Furthermore, the example of Verma modules discussed below shows that $P(V)$ need not be a subset of P in that case.

PROPOSITION 8.2. *Let V be a finite-dimensional representation of the complex semi-simple Lie algebra $\mathfrak{g}$. The set $P(V)$ of weights is contained in the weight lattice P, and is invariant under the action of the Weyl group W. Furthermore,*

$$V = \bigoplus_{\mu \in P(V)} V_\mu.$$

PROOF. Since V is finite-dimensional, Weyl's theorem shows that it is completely reducible as a $\mathfrak{g}$-representation, and also as a $\mathfrak{sl}_\alpha$-representation. That is, it breaks up as a direct sum of irreducible $\mathfrak{sl}_\alpha$-representations, for any given α. In particular, the transformations $\pi(h_\alpha)$, $\alpha \in \mathfrak{R}$ are all diagonalizable. Since these transformations commute, they are in fact *simultanously* diagonalizable. Since the h_α span $\mathfrak{t}$, it follows that V is a direct sum of the weight spaces. Suppose μ is a weight, and let $v \in V_\mu$ be non-zero. Then v is in particular an eigenvector of $\pi(h_\alpha)$, with eigenvalue $\langle \mu, \alpha^\vee\rangle$. By the representation theory of $\mathfrak{sl}_\alpha$, the eigenvalues of $\pi(h_\alpha)$ are integers. This shows $\mu \in P$. For the W-invariance, consider the automorphism

$$\Theta_\alpha = \exp(\pi(e_\alpha))\exp(-\pi(f_\alpha))\exp(\pi(e_\alpha)) \in \mathrm{GL}(V). \tag{122}$$

(Here the finite-dimensionality of V is used to define the exponential of an endomorphism of V.) Using the same argument as in the proof of Proposition 6.1, we see that Θ_α implements w_α:

$$\Theta_\alpha \pi(h) \Theta_\alpha^{-1} = \pi(w_\alpha h), \quad h \in \mathfrak{t}.$$

Hence Θ_α takes V_μ to $V_{w_\alpha(\mu)}$. Since this is true for all $\alpha \in \mathfrak{R}$, $\mu \in P(V)$, this shows the W-invariance of $P(V)$. $\square$

Since $P(V)$ is W-invariant, it is uniquely determined by its intersection with the set

$$P_+ = \{\mu \in P |\ \langle \mu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}\ \forall \alpha \in \mathfrak{R}_+\}$$

of *dominant weights*.

9. Highest weight representations

Let $\mathfrak{n}$ (resp. $\mathfrak{n}_-$) be the nilpotent Lie subalgebra of $\mathfrak{g}$ defined as the direct sum of root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \mathfrak{R}_+$ (resp. $-\alpha \in \mathfrak{R}_+$). Let $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ and $\mathfrak{b}_- = \mathfrak{n}_- \oplus \mathfrak{t}$ be the *Borel subalgebras*.

Definition 9.1. Let V be any $\mathfrak{g}$-representation. A non-zero vector $v \in V$ is called a *highest weight vector*, of highest weight $\mu \in \mathfrak{t}^*$, if

$$v \in V_\mu, \quad \pi(\mathfrak{n})v = 0.$$

The representation V is called a *highest weight representation* if there is a highest weight vector v with $V = \pi(U\mathfrak{g})v$.

The highest weight vectors span the subspace

$$V^{\mathfrak{n}} = \{v \in V |\ \pi(\mathfrak{n})v = 0\}.$$

Note that $V^{\mathfrak{n}}$ is invariant under $U(\mathfrak{t})$, and that it is annihilated by all of $U(\mathfrak{g})\mathfrak{n}$. Using the direct sum decomposition $U(\mathfrak{g}) = U(\mathfrak{n}_-) \oplus U(\mathfrak{g})\mathfrak{b}$, it follows that

$$V = U(\mathfrak{n}_-)V^{\mathfrak{n}}.$$

An important example of a highest weight representation is given as follows. Given $\mu \in \mathfrak{t}^*$, define a representation of $\mathfrak{b}$ on $\mathbb{C}$ by letting $\xi \in \mathfrak{t}$ act as a scalar $\langle \mu, \xi \rangle$ and letting $\mathfrak{n}$ act as zero. Let

$$L(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}$$

be the induced $\mathfrak{g}$-representation (where the $\mathfrak{g}$-action comes from the left regular representation on $U(\mathfrak{g})$). The image $v \in L(\mu)$ of $1 \in U(\mathfrak{g})$ is a highest weight vector, of weight μ. One calls $L(\mu)$ the *Verma module*. It is the universal highest weight module, in the following sense.

Proposition 9.2. *Let V be a highest weight representation, of highest weight $\mu \in \mathfrak{t}^*$. Then there exists a surjective $\mathfrak{g}$-module morphism $L(\mu) \to V$.*

Proof. Let $v \in V$ be a highest weight vector. The map $\mathbb{C} \to V$, $\lambda \mapsto \lambda v$ is $U(\mathfrak{b})$-equivariant, since $\pi(\mathfrak{n})v = 0$, $\pi(\xi)v = \langle \mu, \xi \rangle v$, $\xi \in \mathfrak{t}$. Hence, the surjective $\mathfrak{g}$-map $U(\mathfrak{g}) \to V$ $x \mapsto \pi(x)v$ descends to a surjective $\mathfrak{g}$-map $L(\mu) \to V$. $\square$

Denote by $\mathrm{cone}_{\mathbb{Z}} \mathfrak{R}_+$ all sums $\sum_i k_i \alpha_i$ with integers $k_i \geq 0$.

PROPOSITION 9.3. *The Verma module $V(\mu)$ is a direct sum of its weight spaces. We have*
$$P(V(\mu)) = \mu - \mathrm{cone}_{\mathbb{Z}}\,\mathfrak{R}_+;$$
the multiplicity of the weight $\mu - \nu$ (for $\nu \in \mathrm{cone}_{\mathbb{Z}}\,\mathfrak{R}_+$) is equal to the cardinality of the set of maps $\mathfrak{R}_+ \to \mathbb{Z}_{\geq 0},\ \alpha \mapsto k_\alpha$ such that $\nu = \sum_\alpha k_\alpha \alpha$. In particular, $L(\mu)_\mu$ is 1-dimensional.

PROOF. Choose an ordering of the set $\mathfrak{R}_+$. By the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{n}_-)$ has a basis consisting of ordered products $\prod_{\alpha\in\mathfrak{R}_+} f_\alpha^{k_\alpha}$. Hence $L(\mu)$ has a basis
$$v_{\{k_\alpha\}} = \prod_{\alpha\in\mathfrak{R}_+} \pi(f_\alpha)^{k_\alpha} v$$
where v is the highest weight vector. Since $\pi(f_\alpha)$ shifts weights by $-\alpha$, the basis vectors are weight vectors, of weight
$$\mu - \sum_{\alpha\in\mathfrak{R}_+} k_\alpha \alpha.$$
□

COROLLARY 9.4. *Let V be a highest weight module for $\mu \in \mathfrak{t}^*$. Then $P(V) \subset \mu - \mathrm{cone}_{\mathbb{Z}}\,\mathfrak{R}_+$, and weights have finite multiplicity. The weight μ has multiplicity 1. If V is irreducible, then $V^{\mathfrak{n}} = V_\mu$.*

PROOF. The first part follows since any highest weight module is a quotient of the Verma module by some submodule. Suppose V is irreducible, and suppose μ' is another highest weight. Then $P(V) \subset \mu' - \mathrm{cone}_{\mathbb{Z}}\,\mathfrak{R}_+$. Since μ' itself lies in $\mu - \mathrm{cone}_{\mathbb{Z}}\,\mathfrak{R}_+$, this is impossible unless $\mu' = \mu$. Equivalently, V_μ contains all highest weight vectors. □

The sum of two proper submodules of $L(\mu)$ is again a proper submodule. (Any submodule is a sum of weight spaces; the submodule is proper if and only if μ does not appear as a weight.) Taking the sum of all proper submodules, we obtain a maximal proper submodule $L'(\mu)$. The quotient module
$$V(\mu) = L(\mu)/L'(\mu)$$
is then irreducible. (The preimage of a proper submodule $W \subset V(\mu)$ is a proper submodule in $L(\mu)$, hence contained in $L'(\mu)$. Thus $W = 0$.)

PROPOSITION 9.5. *Let V be an irreducible $\mathfrak{g}$-representation of highest weight $\mu \in \mathfrak{t}^*$. Then V is isomorphic to $V(\mu)$; the isomorphism is unique up to a non-zero scalar.*

PROOF. We will show that if V, V' are two irreducible modules of highest weight $\mu \in \mathfrak{t}^*$, then $V \cong V'$. (Uniquencess of the isomorphism follows since V_μ, V'_μ are 1-dimensional.) Let $v \in V,\ v' \in V'$ be highest weight vectors. Let $S \subset V \oplus V'$ be the subrepresentation generated by $s = v \oplus v'$, that is, $S = U(\mathfrak{n}_-)s$. Since $\mathfrak{n}$ annihilates s, while $h \in \mathfrak{t}$ acts as a scalar $\langle \mu, h\rangle$, the representation S is again a highest weight module for μ. The projection $p\colon S \to V$ is $\mathfrak{g}$-equivariant, and hence is surjective:

$$p(S) = p(U(\mathfrak{g})s) = U(\mathfrak{g})p(s) = U(\mathfrak{g})v = V.$$

We claim that p is also injective. Suppose not, so that $\ker(p) = S \cap (0 \oplus V') \subset S$ is a non-trivial subrepresentation. The restriction of $p'\colon S \to V'$ to $\ker(p)$ is clearly injective. Since V' is irreducible, this restriction is also surjective, hence $\ker(p) \cong V'$ is a highest weight module of highest weight μ. But S_μ is spanned by $s \notin \ker(p)$, hence $\ker(p)_\mu = 0$. This contradiction shows that $\ker(p) = 0$, hence p is an isomorphism. Likewise $p'\colon S \to V'$ is an isomorphism, proving $V \cong V'$. $\square$

Proposition 9.6. *Let V be an irreducible highest weight representation, of highest weight $\mu \in \mathfrak{t}^*$. Then*

$$\dim V < \infty \Leftrightarrow \mu \in P_+.$$

Proof. Suppose $\mu \in P_+$, and let $v \in V_\mu$ be non-zero. Given $\alpha \in \mathfrak{R}_+$, we have $\pi(e_\alpha)v = 0$, $\pi(h_\alpha)v = \langle \mu, \alpha^\vee \rangle v$. Then $v_j = \frac{1}{j!}\pi(f_\alpha)^j v$, $j = 0, 1, 2, \ldots$ span an irreducible $\mathfrak{sl}_\alpha$-representation $W \subset V$. It is finite-dimensional since $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$. The subspace $\pi(\mathfrak{g})W$ is again a finite-dimensional $\mathfrak{sl}_\alpha$-invariant subspace. Indeed, for all $\xi \in \mathfrak{sl}_\alpha$

$$\pi(\xi)\pi(\mathfrak{g})W \subset \pi(\mathfrak{g})W + \pi(\mathfrak{g})\pi(\xi)W \subset \pi(\mathfrak{g})W.$$

By induction, we hence see that $\pi(U^{(r)})\mathfrak{g})v = \pi(U^{(r)}\mathfrak{g})W$ is a finite-dimensional $\mathfrak{sl}_\alpha$-invariant subspace. This shows that *all* vectors $w \in V$ are contained in some finite-dimensional $\mathfrak{sl}_\alpha$-subrepresentation, and hence that the operators $\pi(e_\alpha), \pi(f_\alpha)$ are locally nilpotent. (That is, for all $w \in V$ there exists $N > 0$ such that $\pi(e_\alpha)^N w = 0$ and $\pi(f_\alpha)^N w = 0$.) As a consequence, the transformation Θ_α defined in (122) is a well-defined automorphism of V, with

$$\Theta_\alpha \circ \pi(h) \circ \Theta_\alpha^{-1} = \pi(w_\alpha h)$$

for all $h \in \mathfrak{t}$. It follows that $P(V)$ is w_α-invariant. Since α was arbitrary, this proves that $P(V)$ is W-invariant. But $P(V) \subset \mu - \operatorname{cone} \mathfrak{R}_+$ has compact intersection with P_+. We conclude that $P(V)$ is finite. Since the weights have finite multiplicity, it then follows that $\dim V < \infty$. $\square$

In summary, we have proved the following result:

theorem 9.7 (Finite-dimensional irreducible $\mathfrak{g}$-representations). *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. Then the irreducible $\mathfrak{g}$-representations are classified by the set P_+ of dominant weights. More precisely, any such representation is isomorphic to $V(\mu)$, for a unique $\mu \in P_+$.*

Definition 9.8. Let V be a finite-dimensional $\mathfrak{g}$-representation, and $\mu \in P_+$. The *μ-isotypical subspace* $V_{[\mu]}$ is the direct sum of the irreducible components of highest weight μ.

Thus

$$V = \bigoplus_{\mu \in P_+} V_{[\mu]}.$$

$V_{[\mu]}$ is equivalently characterized as the image of the map

$$\mathrm{Hom}_{\mathfrak{g}}(V(\mu), V) \otimes V(\mu) \to V.$$

or as

$$V_{[\mu]} = \pi(U\mathfrak{n}_-)V_\mu^{\mathfrak{n}}.$$

The multiplicity of the representation $V(\mu)$ equals

$$\dim \mathrm{Hom}_{\mathfrak{g}}(V(\mu), V) = \dim V_\mu^{\mathfrak{n}}.$$

In particular, the number of all irreducible components of V is $\dim V^{\mathfrak{n}}$.

10. Extremal weights

LEMMA 10.1. *Suppose V is an irreducible $\mathfrak{g}$-representation of highest weight $\mu \in P_+$. Then $||\nu + \rho|| \le ||\mu + \rho||$ for all $\nu \in P(V)$, with equality if and only if $\nu = \mu$.*

PROOF. Choose $w \in W$ such that $w^{-1}(\nu+\rho) \in P_+$. Since $w^{-1}\nu \in P(V)$, the difference $\mu - w^{-1}\nu$ lies in the root cone $\mathrm{cone}_{\mathbb{Z}}\, \mathfrak{R}_+$. Similarly $\rho - w^{-1}\rho \in \mathrm{cone}_{\mathbb{Z}}\, \mathfrak{R}_+$ lies in the positive root cone. Hence, both have non-negative inner product with $w^{-1}(\nu + \rho)$. Writing

$$\mu + \rho = (\mu - w^{-1}\nu) + (\rho - w^{-1}\rho) + w^{-1}(\nu + \rho)$$

it follows that $||\mu + \rho||^2 \ge ||w^{-1}(\nu + \rho)||^2 = ||\nu + \rho||^2$. Suppose equality holds. Then

$$(\mu - w^{-1}\nu) + (\rho - w^{-1}\rho) = 0.$$

Since both summands are in the positive root cone, each of them has to vanish, giving $\nu = w\mu$ and $\rho = w\rho$. Since the W-stabilizer of ρ is trivial, this implies $w = 1$ and $\mu = \nu$. □

PROPOSITION 10.2 (Extremal weights). *Let V be a finite-dimensional unitary $\mathfrak{g}$-representation, and $\mu \in P(V)$ with the property*

$$||\nu + \rho|| \le ||\mu + \rho||$$

for all $\nu \in P(V)$. Then μ is a dominant weight, and the irreducible $\mathfrak{g}$-representation of highest weight μ appears in V, with multiplicity equal to the dimension of the $\mathfrak{t}$-weight space $\dim V_\mu$.

PROOF. Decompose V into irreducible components. By the Lemma, if $\mu \in P(V_1)$ for some irreducible component V_1, then it must be the highest weight of V_1, with $\dim((V_1)_\mu) = 1$. Hence, the multiplicity of the μ-representation is equal to $\dim V_\mu$. □

APPENDIX C

Background on Lie groups

In this appendix we review some basic material on Lie groups. Standard references include [**15**] and [**26**].

1. Preliminaries

A (real) *Lie group* is a group G, equipped with a (real) manifold structure such that the group operations of multiplication and inversion are smooth. For example, $\mathrm{GL}(N,\mathbb{R})$, with manifold structure as an open subset of $\mathrm{Mat}_N(\mathbb{R})$, is a obviously Lie group. According to theorem of E. Cartan, any (topologically) closed subgroup H of a Lie group G is a Lie subgroup: the smoothness is automatic. Hence, it is immediate that e.g. that $\mathrm{SO}(n)$, $\mathrm{GL}(N,\mathbb{C})$, $\mathrm{U}(n)$ etc. are again Lie groups. A related result is that if G_1, G_2 are Lie groups, then any continuous group homomorphism $G_1 \to G_2$ is smooth. Consequently, a given topological group cannot carry more than one smooth structure making it into a Lie group.

For $a \in G$, let $\mathcal{A}^L(a)$ be the diffeomorphism of G given by left multiplication, $g \mapsto ag$. A vector field $X \in \mathfrak{X}(G)$ is called *left-invariant* if it is invariant under all $\mathcal{A}^L(a)$, $a \in G$. Equivalently, a vector field is left-invariant if and only if its action on functions commutes with pull-back under $\mathcal{A}^L(a)$, for all a. It is immediate that the Lie bracket of two left-invariant vector fields is again left-invariant. Let $\mathfrak{X}^L(G) \subset \mathfrak{X}(G)$ denote the Lie algebra of left-invariant vector fields. Any element of $\mathfrak{X}^L(G)$ is determined by its value at the group unit $e \in G$. This gives a vector space isomorphism $T_eG \to \mathfrak{X}^L(G)$, $\xi \mapsto \xi^L$. One calls

$$\mathfrak{g} = T_eG \cong \mathfrak{X}^L(G),$$

with Lie bracket induced from that on $\mathfrak{X}^L(G)$, the Lie algebra of G. *Lie's third theorem* asserts that any finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ arises in this way from a Lie group G, and in fact there is a unique connected, simply connected Lie group having $\mathfrak{g}$ as its Lie algebra.

If $G = \mathrm{GL}(N,\mathbb{R})$, the tangent space $\mathfrak{g} = T_eG$ is canonically identified with the space $\mathrm{Mat}_N(\mathbb{R})$ of $N \times N$-matrices, and one may verify that the Lie bracket is simply the commutator of matrices. (This is the main reason for working with $\mathfrak{X}^L(G)$ rather than $\mathfrak{X}^R(G)$, since the latter choice would have produced *minus* the commutator.)

2. Group actions on manifolds

An *action of a Lie group G on a manifold M* is a group homomorphism $\mathcal{A}\colon G \to \mathrm{Diff}(M)$ into the group of diffeomorphisms of M, with the property that the action map $G \times M \to M,\ (g, x) \mapsto \mathcal{A}(g)(x)$ is smooth. It induces actions on the tangent bundle and cotangent bundle, and hence there are notions of invariant vector fields $\mathfrak{X}(M)^G$, invariant differential forms $\Omega(M)^G$ and so on.

EXAMPLE 2.1. There are three important actions of a Lie group on itself: The actions by left-and right-multiplication, and the adjoint action:

$$\mathcal{A}^L(g)(a) = ga,\ \mathcal{A}^R(g)(a) = ag^{-1},\ \ \mathrm{Ad}(g)(a) = gag^{-1}.$$

An *action of a Lie algebra $\mathfrak{g}$ on a manifold M* is a Lie algebra homomorphism $\mathcal{A}\colon \mathfrak{g} \to \mathfrak{X}(M)$ such that the map $\mathfrak{g} \times M \to TM,\ (\xi, x) \mapsto \mathcal{A}(\xi)(x)$ is smooth.

Given a Lie group action

$$\mathcal{A}\colon G \to \mathrm{Diff}(M),$$

its differential at the group unit defines an action of the Lie algebra $\mathfrak{g}$ (which we denote by the same letter). In terms of the actions of vector fields on functions,

$$\mathcal{A}(\xi)f = \frac{\partial}{\partial t}|_{t=0} \exp(-t\xi)^* f, \quad \xi \in \mathfrak{g}.$$

One calls $\xi_M = \mathcal{A}(\xi)$ the generating vector fields[1] for the G-action.

EXAMPLE 2.2. The generating vector fields $\mathrm{Ad}(\xi) \in \mathfrak{X}(\mathfrak{g})$ for the adjoint action of G on $\mathfrak{g}$ are

$$\mathrm{Ad}(\xi)|_\mu = \mathrm{ad}_\mu(\xi)$$

(using the identifications $T_\mu\mathfrak{g} = \mathfrak{g}$). The generating vector fields for the three natural actions of G on itself are

$$\mathcal{A}^L(\xi) = -\xi^R,\ \ \mathcal{A}^R(\xi) = \xi^L,\ \ \mathrm{Ad}(\xi) = \xi^L - \xi^R.$$

(Note that the vector field $\mathcal{A}^R(\xi)$ must be left-invariant, since the action $\mathcal{A}^R(g)$ commutes with the left-action.) We have $[\xi^L, \zeta^R] = 0$, since the left and right actions commute.

The action of G on M lifts to an action on the tangent bundle TM. Given a fixed point $x \in M$ of a G-action, so that $\mathcal{A}(g)x = x$ for all $g \in G$, the action preserves the fiber T_xM, defining a linear representation $G \to \mathrm{GL}(T_xM)$. In particular, the adjoint action

$$\mathrm{Ad}\colon G \to \mathrm{Diff}(G)$$

fixes e, and hence induces a linear action on $T_eG = \mathfrak{g}$, denoted by the same letter:

$$\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g}).$$

[1]Some authors use opposite sign conventions, so that ξ_M is an anti-homomorphism.

Since the adjoint action on G is by group automorphisms, its linearization acts by Lie algebra automorphisms of $\mathfrak{g}$. One also defines an infinitesimal adjoint action,

$$\mathrm{ad}_\mu\colon \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{ad}_\mu(\xi) = [\mu, \xi]_\mathfrak{g}.$$

Then $\mathrm{ad}\colon \mu \mapsto \mathrm{ad}_\mu$ is a Lie algebra homomorphism

$$\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$$

into the Lie algebra of derivations of $\mathfrak{g}$. (A linear map $A \in \mathrm{End}(\mathfrak{g})$ is a derivation of the Lie bracket if and only if $A[\xi_1, \xi_2] = [A\xi_1, \xi_2] + [\xi_1, A\xi_2]$.)

This *adjoint representation* $\mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ of the Lie algebra is the differential of the adjoint representation $G \to \mathrm{Aut}(\mathfrak{g})$ of the Lie group (note that $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathrm{Der}(\mathfrak{g})$).

3. The exponential map

Any $\xi \in \mathfrak{g} = T_eG$ determines a unique 1-parameter subgroup $\phi_\xi\colon \mathbb{R} \to G$ such that

$$\phi_\xi(t_1 + t_2) = \phi_\xi(t_1)\phi_\xi(t_2), \ \phi_\xi(0) = e, \ \frac{\partial \phi_\xi}{\partial t}|_{t=0} = \xi.$$

In fact, ϕ_ξ is a solution curve of the left-invariant vector field ξ^L. One defines the *exponential map*

$$\exp\colon \mathfrak{g} \to G, \ \xi \mapsto \phi_\xi(1).$$

For matrices, the abstract exponential map coincides with the usual exponential of matrices as a Taylor series. The 1-parameter subgroup may be written in terms of the exponential map as $\phi_\xi(t) = \exp(t\xi)$.

The exponential map is natural with respect to Lie morphisms. Hence, if $\phi\colon G \to H$ is a morphism of Lie groups, and denoting by the same letter its differential $\phi\colon \mathfrak{g} \to \mathfrak{h}$ we have $\exp(\phi(\xi)) = \phi(\exp(\xi))$. In particular, this applies to the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ and its differential $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. That is,

$$\mathrm{Ad}(\exp(\mu)) = \exp(\mathrm{ad}_\mu) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{ad}_\mu^n .$$

Since $(\mathrm{d}_0 \exp)(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \exp(t\xi) = \xi$, the differential of the exponential map at the origin is the identity $\mathrm{d}_0 \exp = \mathrm{id}$. Hence, by the implicit function theorem the exponential map gives a diffeomorphism from an open neighborhood of 0 in $\mathfrak{g}$ to an open neighborhood of e in G. We are interested in the differential of $\exp\colon \mathfrak{g} \to G$ at *any* given point $\mu \in \mathfrak{g}$. It is a linear operator $\mathrm{d}_\mu \exp\colon \mathfrak{g} = T_\mu\mathfrak{g} \to T_gG$. Since $\mathfrak{g}$ is a vector space, $T_\mu\mathfrak{g} \cong \mathfrak{g}$ canonically. On the other hand, we may use the left-action to obtain an isomorphism, $\mathrm{d}_e\mathcal{A}^L(g)\colon \mathfrak{g} \to T_gG$, and hence an isomorphism $TG = G \times \mathfrak{g}$ by *left-trivialization*.

THEOREM 3.1. *The differential of the exponential map* $\exp\colon \mathfrak{g} \to G$ *at* $\mu \in \mathfrak{g}$ *is the linear operator* $d_\mu \exp\colon \mathfrak{g} \to T_{\exp(\mu)}\mathfrak{g}$ *given by the formula,*

$$d_\mu \exp = j^L(\mathrm{ad}_\mu)$$

where we use left-trivialization to identify $T_{\exp(\mu)}\mathfrak{g} \cong \mathfrak{g}$.

Here $j^L(z) = \frac{1-e^{-z}}{z}$ is the holomorphic function introduced in IV.3.5.

PROOF. The differential $\mathrm{d}_\mu \exp(\zeta) = \frac{\partial}{\partial t}\big|_{t=0} \exp(\mu + t\zeta)$ may be written,

$$\mathrm{d}_\mu \exp(\zeta) = d_e(\mathcal{A}^L(\exp(\mu)))\ \frac{\partial}{\partial t}\Big|_{t=0}\Big(\exp(\mu)^{-1}\exp(\mu + t\zeta)\Big)$$

Let $\exp_s(\nu) := \exp(s\nu)$ and (for any given μ, ζ)

$$\phi(s,t) = \exp_s(\mu)^{-1}\exp_s(\mu + t\zeta).$$

Write $\psi(s) = \frac{\partial \phi}{\partial t}|_{t=0} \in \mathfrak{g}$. Thus $\psi(1) = \mathrm{d}_\mu \exp(\zeta)$, while $\psi(0) = 0$ since $\phi(0,t) = e$ for all t. Taking the t-derivative of the equation

$$\mu + t\zeta = \frac{\partial}{\partial s}\Big(\exp_s(\mu + t\zeta)\Big)\ \exp_s(\mu + t\zeta)^{-1} = \frac{\partial}{\partial s}\Big(\exp_s(\mu)\phi\Big)\ \phi^{-1}\exp_s(\mu)^{-1}$$

at $t = 0$, we obtain

$$\begin{aligned}
\zeta &= \frac{\partial}{\partial s}\Big(\exp_s(\mu)\psi\Big)\ \exp_s(\mu)^{-1} - \frac{\partial}{\partial s}\big(\exp_s(\mu)\big)\ \psi\exp_s(\mu)^{-1}\\
&= \exp_s(\mu)\frac{\partial \psi}{\partial s}\exp_s(\mu)^{-1}\\
&= \mathrm{Ad}(\exp_s(\mu))\frac{\partial \psi}{\partial s}\\
&= \exp(s\,\mathrm{ad}_\mu)\frac{\partial \psi}{\partial s}.
\end{aligned}$$

That is, $\frac{\partial \psi}{\partial s} = \exp(-s\,\mathrm{ad}_\mu)\zeta$. Integrating,

$$\psi(1) = \Big(\int_0^1 \exp(-s\,\mathrm{ad}_\mu)\mathrm{d}s\Big)\,\zeta = \frac{1-\exp(-\mathrm{ad}_\mu)}{\mathrm{ad}_\mu}\zeta = j^L(\mathrm{ad}_\mu)\zeta. \qquad \square$$

REMARKS 3.2. (1) Using instead the right action to identify $TG \cong G \times \mathfrak{g}$ one obtains

$$\mathrm{d}_\mu \exp = j^R(\mathrm{ad}_\mu)$$

This follows from the formula for the left trivialization, because the adjoint action of $\exp\mu$ on $\mathfrak{g}$ is

$$\mathrm{Ad}(\exp\mu) = \mathrm{d}_e\mathcal{A}^R(\exp\mu)^{-1} \circ \mathrm{d}_a\mathcal{A}^L(\exp\mu),$$

and since

$$\mathrm{Ad}(\exp\mu)j^L(\mathrm{ad}_\mu) = e^{\mathrm{ad}_\mu}j^L(\mathrm{ad}_\mu) = j^R(\mathrm{ad}_\mu).$$

(2) In particular, the Jacobian of the exponential map relative to the left-invariant volume form is the function, $\mu \mapsto \det(j^L(\mathrm{ad}_\mu))$. while for the right-invariant volume form one obtains $\det(j^R(\mathrm{ad}_\mu))$. In general, the two Jacobians are not the same: Their quotient is the function
$$\det(e^{\mathrm{ad}_\mu}) = e^{\mathrm{tr}(\mathrm{ad}_\mu)}.$$
The function $G \to \mathbb{R}^\times$, $g \mapsto \det(\mathrm{Ad}(g))$ is a group homomorphism called the unimodular character; it relates the left- and right-invariant volume forms Γ^L and Γ^R on G defined by a generator $\Gamma \in \det(\wedge \mathfrak{g}^*)$. The map $\mathfrak{g} \to \mathbb{R}$, $\mu \mapsto \mathrm{tr}(\mathrm{ad}_\mu)$ is a Lie algebra homomorphism called the (infinitesimal) unimodular character. A Lie group is called unimodular if the unimodular character is trivial. For instance, any compact Lie group, and any semi-simple Lie group, is unimodular. The simply connected Lie group corresponding to the non-trivial 2-dimensional Lie algebra is not unimodular.

If G is connected and $\mathfrak{g}$ is quadratic (i.e. it admits an Ad-invariant quadratic form), then G is unimodular. This follows because in that case, ad_μ is skew-adjoint, so its trace vanishes. In the quadratic case, the determinants of $j^L(\mathrm{ad}_\mu)$ and $j^R(\mathrm{ad}_\mu)$ coincide, and are equal to
$$J(\mu) := \det j(\mathrm{ad}_\mu) = \det\Big(\frac{\sinh \mathrm{ad}_\mu/2}{\mathrm{ad}_\mu/2}\Big).$$
By our results from §4 3.7 this function admits a global analytic square root.

4. The vector field $\frac{1}{2}(\xi^L + \xi^R)$

Any $\xi \in \mathfrak{g}$ may be viewed as a constant vector field on $\mathfrak{g}$. The half-sum $\xi^\sharp = \frac{1}{2}(\xi^L + \xi^R) \in \mathfrak{X}(G)$ is the closest counterpart of the constant vector field $\xi \in \mathfrak{X}(\mathfrak{g})$. For example, the vector fields $\xi^\sharp$ 'almost' commute in the sense that
$$[\xi^\sharp, \zeta^\sharp] = \frac{1}{4}[\xi,\zeta]^L - \frac{1}{4}[\xi,\zeta]^R = \frac{1}{4}\,\mathrm{Ad}([\xi,\zeta])$$
vanishes at $e \in G$. Note also that
$$[\mathrm{Ad}(\xi), \zeta^\sharp] = [\xi,\zeta]^\sharp,$$
parallel to a property of the constant vector field on $\mathfrak{g}$.

Let $\mathfrak{g}' \subset \mathfrak{g}$ denote the subset where the exponential map has maximal rank. By the formula for $\mathrm{d}_\mu \exp$, this is the subset where $\mathrm{ad}_\mu \colon \mathfrak{g} \to \mathfrak{g}$ has no eigenvalue of the form $2\pi\sqrt{-1}k$ with $k \in \mathbb{Z} - \{0\}$. Given a vector field $X \in \mathfrak{X}(G)$, one has a well-defined vector field $\exp^*(X) \in \mathfrak{X}(\mathfrak{g}')$ such that $\exp^*(X)_\mu = (\mathrm{d}_\mu \exp)^{-1}(X_{\exp \mu})$ for all $\mu \in \mathfrak{g}^*$. In particular, for $\xi \in \mathfrak{g}$ we can consider
$$\exp^* \xi^L,\ \exp^* \xi^R,\ \exp^* \xi^\sharp.$$

Since $T_\mu\mathfrak{g} \cong \mathfrak{g}$, both of these vector fields define elements of $C^\infty(\mathfrak{g}') \otimes \mathfrak{g}$, depending linearly on ξ. The map taking ξ to this vector field is therefore an element of $C^\infty(\mathfrak{g}^*) \otimes \operatorname{End}(\mathfrak{g})$.

Using left-trivialization of the tangent bundle, we have

$$(\exp^* \xi^L)_\mu = (j^L(\mathrm{ad}_\mu))^{-1}(\xi) = \frac{\mathrm{ad}_\mu}{1 - e^{-\mathrm{ad}_\mu}}\xi$$

Similarly,

$$(\exp^* \xi^R)_\mu = (j^R(\mathrm{ad}_\mu))^{-1}(\xi) = \frac{\mathrm{ad}_\mu}{e^{\mathrm{ad}_\mu} - 1}\xi.$$

The difference with the constant vector field ξ is,

$$\begin{aligned}(\exp^* \xi^L)_\mu - \xi &= \mathrm{ad}_\mu f^L(\mathrm{ad}_\mu)(\xi) = \mathrm{Ad}(f^L(\mathrm{ad}_\mu)\xi),\\ (\exp^* \xi^R)_\mu - \xi &= \mathrm{ad}_\mu f^R(\mathrm{ad}_\mu)(\xi) = \mathrm{Ad}(f^R(\mathrm{ad}_\mu)\xi).\end{aligned}$$

where

$$f^L(z) = \frac{1}{1-e^{-z}} - \frac{1}{z}, \quad f^R(z) = \frac{1}{e^z - 1} - \frac{1}{z}.$$

Note that $f^L(\mathrm{ad}_\mu), f^R(\mathrm{ad}_\mu) \in \operatorname{End}(\mathfrak{g})$ are well-defined for all $\mu \in \mathfrak{g}'$. The formula shows that the difference between the vector fields $\exp^* \xi^L, \exp^* \xi^R$ and the constant vector field ξ is a vector field in the direction of the orbits of the adjoint action. Put differently, the radial part of these vector fields equals ξ. Finally,

$$(\frac{1}{2}\exp^*(\xi^L + \xi^R))_\mu - \xi = f(\mathrm{ad}_\mu)(\mathrm{ad}_\mu \xi) = \mathrm{Ad}\left(f(\mathrm{ad}_\mu)\xi\right)\big|_\mu,$$

where $f = \frac{1}{2}(f^L + f^R)$. That is,

$$f(z) = \frac{1}{2}\Big(\frac{1}{e^z-1} + \frac{1}{1-e^{-z}}\Big) - \frac{1}{z} = \frac{1}{2}\coth\big(\frac{z}{2}\big) - \frac{1}{z}.$$

Remarks 4.1. The function $j^R(z)^{-1} = \frac{z}{e^z-1}$ is the well-known generating functions for the *Bernoulli numbers* B_n:

$$j^R(z)^{-1} = \frac{z}{e^z-1} = \sum_{n=0}^\infty \frac{B_n}{n!}\, z^n$$

The expansion of the function $f(z)$ reads,

$$f(z) = \frac{1}{2}\coth(\frac{z}{2}) - \frac{1}{z} = \sum_{n=1}^\infty \frac{B_{2n}}{(2n)!} z^{2n-1}.$$

5. Maurer-Cartan forms

The left-invariant Maurer-Cartan form $\theta^L \in \Omega^1(G) \otimes \mathfrak{g}$ is defined in terms of its contractions with left-invariant vector fields by

$$\iota(\xi^L)\theta^L = \xi$$

Similarly, one defines the right-invariant Maurer-Cartan form $\theta^R \in \Omega^1(G)^R \otimes \mathfrak{g}$ by

$$\iota(\xi^R)\theta^R = \xi.$$

For matrix Lie groups, one has the formulas

$$\theta^L = g^{-1}\mathrm{d}g,\ \theta^R = \mathrm{d}g\ g^{-1}.$$

(More precisely, $\mathrm{d}g$ is a matrix-valued 1-form on G, to be interpreted as the pull-back of the coordinate differentials on $\mathrm{Mat}_N(\mathbb{R}) \cong \mathbb{R}^{N^2}$ under the inclusion map $G \to \mathrm{Mat}_N(\mathbb{R})$.)

PROPOSITION 5.1 (Properties of Maurer-Cartan forms). (1) *The Maurer-Cartan forms are related by*

$$\theta^R_g = \mathrm{Ad}_g(\theta^L_g),$$

(2) *The differential of* θ^L, θ^R *is given by the* Maurer-Cartan equations

$$d\theta^L + \tfrac{1}{2}[\theta^L, \theta^L] = 0, \quad d\theta^R - \tfrac{1}{2}[\theta^R, \theta^R] = 0.$$

(3) *The pull-backs of* θ^L, θ^R *under group multiplication* $\mathrm{Mult}\colon G \times G \to G, (g_1, g_2) \mapsto g_1 g_2$ *are given by the formula,*

$$\mathrm{Mult}^*\,\theta^L = \mathrm{Ad}_{g_2^{-1}}\,\mathrm{pr}_1^*\,\theta^L + \mathrm{pr}_2^*\,\theta^L,$$
$$\mathrm{Mult}^*\,\theta^R = \mathrm{Ad}_{g_1}\,\mathrm{pr}_2^*\,\theta^R + \mathrm{pr}_1^*\,\theta^L.$$

where $\mathrm{pr}_1, \mathrm{pr}_2\colon G \times G \to G$ *are the two projections.*

For matrix Lie groups, all of these results are easily proved from $\theta^L = g^{-1}\mathrm{d}g$ and $\theta^R = \mathrm{d}gg^{-1}$. For instance, $\mathrm{Mult}^*\,\theta^L$ is computed as follows:[2]

$$(g_1g_2)^{-1}\mathrm{d}(g_1g_2) = g_2^{-1}g_1^{-1}\mathrm{d}g_1g_1^{-1} + g_2^{-1}\mathrm{d}g_2.$$

Consider now the pull-back of the Maurer-Cartan forms under the exponential map, $\exp^*\theta^L, \exp^*\theta^R \in \Omega^1(\mathfrak{g}) \otimes \mathfrak{g}$. At any given point $\mu \in \mathfrak{g}$, these are elements of $T^*_\mu\mathfrak{g} \otimes \mathfrak{g} = \mathfrak{g}^* \to \mathfrak{g}$. Thus, we can view $\exp^*\theta^L, \exp^*\theta^R$ as maps $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

THEOREM 5.2. *The maps* $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *determined by* $\exp^*\theta^L, \exp^*\theta^R$ *are given by*

$$\mu \mapsto j^L(\mathrm{ad}_\mu), \quad \mu \mapsto j^R(\mathrm{ad}\,\mu),$$

respectively.

[2]For general groups, use that for all $\xi, \zeta \in \mathfrak{g}$, the vector field $\xi_1^R + \zeta_2^L \in \mathfrak{X}(G \times G)$ (where the subscripts indicate the respective G-factor) is Mult-related to $\xi^R + \zeta^L$. Hence $\iota(\xi_1^R + \zeta_2^L)\,\mathrm{Mult}^*\,\theta^L = \mathrm{Mult}^*\,\iota(\xi^R + \zeta^L)\theta^L$.

PROOF. Let $\mu \in \mathfrak{g}$ and $\xi \in T_\mu\mathfrak{g} = \mathfrak{g}$. Then, using left-trivialization of the tangent bundle,

$$\begin{aligned}\iota(\xi)(\exp^* \theta^L)_\mu &= \iota(\mathrm{d}_\mu \exp(\xi))\theta^L_{\exp \mu}\\ &= \mathrm{d}_\mu \exp(\xi)\\ &= j^L(\mathrm{ad}_\mu)(\xi).\end{aligned}$$

For θ^R, just use $\theta^R = \mathrm{Ad}_g\, \theta^L$. □

In a basis $e_i \in \mathfrak{g}$, the Maurer-Cartan forms can be written $\theta^L = \sum_i \theta^{L,i} \otimes e_i$. Letting μ^i be the coordinate functions on $\mathfrak{g}$ and $\mathrm{d}\mu^i$ their differentials, the Theorem says that

$$\exp^* \theta^{L,i} = \sum_j j^L(\mathrm{ad}_\mu)^i_j\, \mathrm{d}\mu^j$$

where $j^L(\mathrm{ad}_\mu)^i_j$ are the components of the matrix describing $j^L(\mathrm{ad}_\mu)$. Dropping indices, we may write this as $\exp^* \theta^L = j^L(\mathrm{ad}_\mu)(\mathrm{d}\mu)$, where $\mathrm{d}\mu \in \Omega^1(\mathfrak{g}) \otimes \mathfrak{g}$ is the tautological 1-form.

The half-sum $\frac{1}{2}(\theta^L + \theta^R)$ is a natural counterpart of $\mathrm{d}\mu \in \Omega^1(\mathfrak{g}; \mathfrak{g})$. The Theorem shows

COROLLARY 5.3.

$$\tfrac{1}{2} \exp^*(\theta^L + \theta^R) - d\mu = \frac{\sinh(\mathrm{ad}_\mu) - \mathrm{ad}_\mu}{\mathrm{ad}_\mu}(d\mu) = g(\mathrm{ad}_\mu)\, \mathrm{ad}_\mu(d\mu)$$

where $g(z) = z^{-2} \frac{\sinh z - z}{z^2}$ *is the function introduced in IV.3.5.*

6. Quadratic Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A bilinear form B on $\mathfrak{g}$ is called G-invariant if it is invariant under the adjoint action:

$$B(\mathrm{Ad}_g(\xi), \mathrm{Ad}_g(\zeta)) = B(\xi, \zeta)$$

for all ξ, ζ. If B is furthermore non-degenerate, we will refer to G as a *quadratic Lie group*. For instance, any semi-simple Lie group is a quadratic Lie group, taking B to be the Killing form. The group $G = \mathrm{GL}(N, \mathbb{R})$ is quadratic, using the trace form $B(\xi, \zeta) = \mathrm{tr}(\xi\zeta)$ on its Lie algebra.

If H is any Lie group, let $G = H \ltimes \mathfrak{h}^*$, with Lie algebra $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{h}^*$ be the semi-direct product. That is, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$ as a vector space, with bracket relations

$$[\xi_1 \oplus \mu_1, \xi_2 \oplus \mu_2] = [\xi_1, \xi_2] \oplus (-\,\mathrm{ad}^*_{\xi_1} \mu_2 + \mathrm{ad}^*_{\xi_2} \mu_1).$$

Then the bilinear form given by the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$ is invariant, hence $H \ltimes \mathfrak{h}^*$ is a quadratic Lie group.

Given an invariant symmetric bilinear form B on $\mathfrak{g}$ (not ncecessarily non-degenerate), one can construct an important 3-form on the group, sometimes called the *Cartan 3-form*:

$$\eta = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) \in \Omega^3(G)$$

PROPOSITION 6.1. *The 3-form η is closed: $d\eta = 0$. Hence it defines a de Rham cohomology class $[\eta] \in H^3(G, \mathbb{R})$.*

PROOF. Using the Maurer-Cartan-equation $\mathrm{d}\theta^L + \frac{1}{2}[\theta^L, \theta^L] = 0$, we have

$$\mathrm{d}\eta = -\frac{1}{24} B([\theta^L, \theta^L], [\theta^L, \theta^L]) = -\frac{1}{24} B(\theta^L, [\theta^L, [\theta^L, \theta^L]]).$$

But $[\theta^L, [\theta^L, \theta^L]] = 0$ by the Jacobi identity for $\mathfrak{g}$. $\square$

The pull-back of $\exp^* \eta$ of the closed 3-form η to $\mathfrak{g}$ is exact. In fact, the Poincaré lemma gives an explicit primitive $\varpi \in \Omega^2(\mathfrak{g})$ with $\mathrm{d}\varpi = \Phi^* \eta$.

Suppose B is non-dgenerate. The identification

$$\Omega^2(\mathfrak{g}) \cong C^\infty(\mathfrak{g}) \otimes \wedge^2 \mathfrak{g}^* \cong C^\infty(\mathfrak{g}) \otimes \mathfrak{o}(\mathfrak{g}).$$

takes ϖ to a function $\mathfrak{g} \to \mathfrak{o}(\mathfrak{g})$.

PROPOSITION 6.2. *The function $\mathfrak{g} \to \mathfrak{o}(\mathfrak{g})$ corresponding to the 2-form ϖ is $\mu \mapsto g(\mathrm{ad}_\mu)$, where $g(z) = z^{-2}(\sinh(z) - z)$.*

PROOF. Recall that the transgression of a form on a vector space V is given by the pull-back under $H \colon I \times V \to V$, $(t, x) \mapsto tx$ followed by integration over the fibers over the projection $\mathrm{pr}_2 \colon I \times X \to X$. We have,

$$\exp^* \theta^L = j^L(\mathrm{ad}_\xi)\mathrm{d}\xi,$$

hence

$$\begin{aligned} H^* \exp^* \theta^L &= j^L(t\, \mathrm{ad}_\xi)(t\mathrm{d}\xi + \xi \mathrm{d}t) \\ &= t\, j^L(t\, \mathrm{ad}_\xi)\mathrm{d}\xi + \xi \mathrm{d}t \end{aligned}$$

Consequently

$$\mathrm{d}H^* \exp^* \theta^L = (1 - \exp(-t\, \mathrm{ad}_\xi))\mathrm{d}\xi \wedge \mathrm{d}t.$$

Hence

$$\begin{aligned} H^* exp^* \eta &= \frac{1}{4} B\big(tj^L(\mathrm{ad}_{t\xi})\mathrm{d}\xi,\ (1 - \exp(-t\, \mathrm{ad}_\xi))\mathrm{d}\xi \wedge \mathrm{d}t\big) \\ &= \frac{1}{4} B(\mathrm{d}\xi,\ tj^R(\mathrm{ad}_{t\xi})(1 - \exp(-t\, \mathrm{ad}_\xi))\mathrm{d}\xi \wedge \mathrm{d}t\big) \\ &= \frac{1}{2} B(\mathrm{d}\xi,\ \frac{\cosh(t\, \mathrm{ad}_\xi) - 1}{\mathrm{ad}_\xi} \mathrm{d}\xi\big)\ \mathrm{d}t \end{aligned}$$

Integrating from $t = 0$ to 1, we get

$$(\mathrm{pr}_2)_* H^* exp^* \eta = \frac{1}{2} B(\mathrm{d}\xi,\ \frac{\sinh(\mathrm{ad}_\xi) - \mathrm{ad}_\xi}{\mathrm{ad}_\xi} \mathrm{d}\xi\big).$$ $\square$

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